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Large scale chaotic motion of charged particles in a longitudinal electrostatic wave

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Abstract. The large scale chaotic motion of a charged particle in a homogeneous static magnetic field and a longitudinal electrostatic wave is discussed. A formula for estimating the stochasticity threshold α_{thr} of the wave amplitude from the wave frequency ν and the propagation angle φ is proposed. For $\varphi = \pi/2$ and $\varphi = \pi/4$ this formula reduces to known results for these special cases. It is shown that the essential characteristics of the $\varphi = \pi/2$ case persist for angles $\pi/2 > \varphi > \pi/3$.

1. Introduction

It is, by now, a well established fact that the majority of time independent Hamiltonian systems with more than one degree of freedom possess trajectories that can be classified by their behaviour in phase space as ordered or disordered (chaotic), and that the same type of system may behave sometimes as integrable (with ordered trajectories and regular invariant curves reflecting the foliation of phase space) and sometimes as non-integrable (with chaotic trajectories and 'dissolved' invariant curves), depending on the value of a parameter that measures the relative strength of the nonlinear force (Helleman 1980). Early papers (e.g. Contopoulos 1963, Hénon and Heiles 1964) concentrated on the ordered, quasi-integrable aspect of dynamical systems. More recently, after ideas on nonlinear dynamics and on generic non-integrability had been more or less crystallised, the interest shifted to the systematic investigation of the chaotic aspect of dynamical systems. Plasma physics is a particular field where the transition to chaos by various nonlinear processes received extensive attention (Treve 1978, Laval and Gresillon 1979). Since wave–particle interactions most often control nonlinear plasma behaviour, the nonlinear interaction of a single charged particle with a large amplitude wave is of paramount importance. It is the purpose of this paper to address some aspects of the large scale chaotic motion of a charged test particle in a homogeneous static magnetic field and a longitudinal electrostatic wave. This dynamical system is characterised by three dimensionless parameters, the wave amplitude α, the wave frequency ν and the propagation angle of the wave (with respect to the magnetic field) φ. Notice for future reference that if any of these parameters is zero, the system becomes integrable and does not possess chaotic trajectories. In the rest of this paper we consider only systems with $\nu \gg 1$.

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Particular cases of this problem already discussed in the literature are the cases of oblique and perpendicular wave propagation ($\widehat{kB_0} \simeq \pi/4$ and $\widehat{kB_0} = \pi/2$ respectively). The first case was discussed by Smith and Kaufman (1975, 1978) and the second by e.g. Karney and Bers (1977), Fukuyama *et al* (1977), Karney (1978, 1979) and Hsu (1982). The increased interest in the perpendicular case is due to the fact that in many cases of interest the plasma waves propagate at angles close to $\pi/2$. Therefore it is important to determine the range of propagation angles about $\pi/2$ that are reasonably well approximated by the exact perpendicular case as far as stochasticity effects are concerned. This point has been discussed by Lichtenberg (1979) and Abe *et al* (1980) and recently by Singh *et al* (1983) and is the main topic of this paper.

One remarkable property of the perpendicular case is that the 'stochasticity threshold' α_{thr} (the value of the wave amplitude above which chaotic trajectories dominate a region of phase space) does not depend sensitively on the number (one or two) of primary resonances of the corresponding dynamical system. We use this fact in the present work to propose a 'stochasticity criterion' that depends on the amplitude, frequency and propagation angle of the wave. We calculate initially the stochasticity criterion for a system with two primary resonances, using the well known method of overlapping resonances (Chirikov 1979) for which calculations are straightforward (e.g. Ford 1978, Greene 1980). Then we make the assumption (ansatz) that this result is valid for a system with one primary resonance as well, and we use it to find the range of angles $\varphi \neq \pi/2$ for which the properties of the wave–particle system (irrespectively of the number of primary resonances) are similar to the properties of the exact perpendicular ($\varphi = \pi/2$) propagation case. Note that in the special cases $\varphi = \pi/2$ and $\varphi \simeq \pi/4$ the above criterion reduces to the results of Fukuyama *et al* (1977) and Smith and Kaufman (1978) respectively.

The paper is organised as follows. In § 2 we formulate the problem and write a dimensionless Hamiltonian function of the form $H = H_0(\bar{I}) + H_1(\bar{I}, \bar{\theta})$. In § 3 we derive a criterion for the onset of chaos and show that for a range of angles around $\pi/2$ the resulting quasi-perpendicular dynamical system has the same properties as the perpendicular one. For propagation angles outside this range (we identify this regime as the oblique case) a different criterion for the onset of chaos is effective, which generally is satisfied for lower wave amplitude values. Finally in § 4 we summarise our results and make some concluding remarks.

2. Formulation of the problem

Consider a single particle of mass m and charge q moving in a static homogeneous magnetic field $\bar{B}_0 = B_0\hat{z}$ and in a longitudinal electrostatic wave $\Phi = -\Phi_0 \cos(k_\perp x + k_\parallel z - \omega t)$ propagating at an angle φ to the magnetic field. The motion of the particle is given by Hamilton's equations

$$\frac{\mathrm{d}q_a}{\mathrm{d}t} = \frac{\partial H}{\partial p_a}, \qquad \frac{\mathrm{d}p_a}{\mathrm{d}t} = -\frac{\partial H}{\partial q_a}, \qquad a = x, y, z \tag{1a}$$

and the Hamiltonian function

$$H = (2m)^{-1}[p_x^2 + (p_y - qxB_0)^2 + p_z^2] - q\Phi_0 \sin(k_\perp x + k_\parallel z - \omega t) \tag{1b}$$

where $xB_0 \equiv A_y$ is the solution of Maxwell's equation $\bar{B} = \nabla \times \bar{A}$. The dynamical system (1) has an obvious integral of motion, $p_y = p_{y0}$ (since y is an ignorable coordinate of (1*b*)). By a shift in the origin of phase space this integral can always be written as $p_{y0}=0$, so without loss of generality we may take p_y to be zero. Changing the time variable according to $\omega t' = -k_\parallel z + t$ (wave frame of reference) and normalising length to $k_\perp^{-1}$ and time to $(qB_0/m)^{-1}$, the Hamiltonian function (1*b*) can be written in action (I_1, I_2) and angle (θ_1, θ_2) variables as

$$H = I_1 + \nu I_2 + \tfrac{1}{2}\xi^2 I_2^2 - \alpha \sum_{l=-\infty}^{\infty} J_l(r)\sin(l\theta_1 - \theta_2) \tag{2}$$

where $r = (2I_1)^{1/2}$ and $\xi = \cot\varphi = k_\parallel/k_\perp \equiv |\bar{k}\cdot\bar{B}_0|/|\bar{k}\times\bar{B}_0|$. Notice that the Hamiltonian (2) is isoenergetically non-degenerate and therefore satisfies the conditions of the KAM theorem. As $\xi \to 0$ it reduces to

$$H = I_1 + \nu I_2 - \alpha \sum_{l=-\infty}^{\infty} J_l(r)\sin(l\theta_1 - \theta_2) \tag{3}$$

that gives the test particle's motion in an exactly perpendicularly ($k_\parallel = 0$) propagating wave (8). A system with $\xi \neq 0$ but with behaviour similar to the $\xi = 0$ case will be referred to in this paper as 'quasi-perpendicular' for obvious reasons. The behaviour of dynamical systems of the type of (2) and (3) for increasing values of α and the study of the transition to chaos by means of surface of section plots and theoretical calculations is beyond the scope of this work and can be found in the references already cited.

Let $H = H_0 + \varepsilon H_1$, $\varepsilon \ll 1$, be a non-integrable Hamiltonian function, where H_0 is an integrable Hamiltonian (Arnold 1978). Then it is known that the method for calculating the stochasticity threshold of the 'perturbation' ε depends on whether H_0 is degenerate or not and whether it possesses one or two lower-order resonances (Codaccioni *et al* 1982). For a non-degenerate

$$\left(\left| \frac{\partial^2 H_0}{\partial I_i \partial I_j} \right| \neq 0 \quad \text{or} \quad \begin{vmatrix} \partial^2 H_0/\partial I_i\,\partial I_j & \partial H_0/\partial I_i \\ \partial H_0/\partial I_j & 0 \end{vmatrix} \neq 0 \right.$$

(Arnold 1978)) one-resonance system the stochasticity threshold can be estimated from the steep growth of the stochastic layer (Codaccioni *et al* 1982) while for a degenerate one-resonance system it can be estimated from the overlapping of the closest higher-order resonances (Fukuyama *et al* 1977). For a non-degenerate two-resonance system the appropriate method is the criterion of primary resonance overlapping (e.g. Ford 1978), while in this paper we propose a method for calculating a criterion in the case of a degenerate two-resonance system.

The dynamical system of (3) has two lower-order resonances for $\nu \simeq n + \frac{1}{2}$, where n is a natural number, and one lower-order resonance for all other values of ν; when $\nu \simeq n$ this resonance is in addition a primary one (following the terminology of Codaccioni *et al*). In the case of (2) the situation is not so simple, because one more parameter has to be considered, the energy of the system; however, quite generally one can say that for $\xi \to 0$ the situation is similar to (3), while for large ξ values the two-resonance case begins to dominate. A remarkable property of (3) is that if $\nu \gg 1$ the large scale stochasticity threshold α_{thr} of the wave amplitude does not depend sensitively on the mechanism through which stochasticity sets in; actually to lowest order the stochasticity criterion turns out to be the same for either a one- or a two-resonance system. This was first observed by Karney (1978) and discussed extensively by Lichtenberg (1979), who attributed this behaviour to the fact that although in a one-resonance system the stochasticity is due to the interaction of secondary islands, this happens when the primary islands have grown considerably, and therefore the onset of stochasticity is essentially controlled by the primary island size, exactly as in the two-resonance case. These ideas have been recently discussed again by Codaccioni *et al* (1982).

In this work we initially calculate the stochasticity criterion for a non-degenerate two-resonance dynamical system ($\nu^* = \nu + \xi^2 I_2 \simeq n^* + \frac{1}{2}$ in (2)) using Chirikov's method of overlapping resonances. This criterion can then be used:

(a) To calculate the stochasticity criterion of a degenerate two-resonance dynamical system ($\nu \simeq n + \frac{1}{2}$ in (3)). As we know that the mechanism of stochasticity onset is the overlapping of two lower-order resonances, we can do this by simply taking the limit of the already calculated criterion for $\xi \to 0$.

(b) To find a 'universal' stochasticity criterion, valid for all values of ξ and ν^*. To do this we invoke the fact that for degenerate systems (3) the stochasticity criterion turns out to be the same—to lowest order—for all ν, and we assume that, in the same way, our criterion is actually valid for all ν^* and ξ. This 'ansatz' is then tested by comparing results derived from this 'universal' criterion with computer experiments.

3. The stochasticity criterion

To find a criterion for the onset of chaotic behaviour in the trajectories of the Hamiltonian (2) in the case of two lower resonances the following information is needed: the two island families whose interaction we assume that causes the chaos, the coordinates of the centres of these islands and their widths, all as functions of the perturbation strength. The island families in the case of two lower-order resonances in (2) can be found by means of a canonical transformation. In appendix 1 we explicitly construct this transformation in a series form that gives the new Hamiltonian

$$H^* = I_1^* + \nu I_2^* + \tfrac{1}{2}\xi^2 I_2^{*2} - \alpha[J_{n^*}(r^*)\sin(n^*\theta_1^* - \theta_2^*) + J_{n^*+1}(r^*)\sin((n^*+1)\theta_1^* - \theta_2^*)] + \ldots \tag{4}$$

where $\nu^* = \nu + \xi^2 I_2$, and $\nu^* \simeq n^* + \frac{1}{2}$. The new canonical variables I_1^*, θ_1^*, I_2^*, θ_2^* are related to the old ones through a near-identity transformation and in appendix 1 it is shown that

$$(I_1^* - I_1)/I_1 \lesssim \alpha\nu^{*-4/3}$$

so that for $\nu^* \gg 1$ we may take $I_1^* \simeq I_1$ in (4). For all values of I_2 then that satisfy the relation $\nu^* \simeq n^* + \frac{1}{2}$ equation (4) can be approximated by the resonance Hamiltonian

$$H_{\mathrm{R}} = I_1 + \nu I_2 + \tfrac{1}{2}\xi^2 I_2^2 - \alpha[J_{n^*}(r)\sin(n^*\theta_1 - \theta_2) + J_{n^*+1}(r)\sin((n^*+1)\theta_1 - \theta_2)]. \tag{5}$$

Following standard procedures (Ford 1978, Greene 1980) for the application of the resonance overlap criterion, we estimate in appendix 2 that chaotic motion sets in whenever

$$\{\alpha|J_{n^*}(r_0)|[\xi^2/n^{*2} + (\alpha/r_0^2)|J''_{n^*}(r_0)|]\}^{1/2} \geqslant \nu^*/4n^*(n^*+1) \simeq 1/4n^* \tag{6}$$

where r_0 is a local maximum of the LHS of (6) and is approximately given by $J'_{n^*}(r_0)=0$. Equation (6) is derived under the condition $\xi^2/n^{*2} > \alpha/r_0^2$ (non-degeneracy condition) and its validity for $\xi=0$ is not obvious. Since, however, it has been proved (Hsu 1982) that in the case $\xi=0$ and $\nu=n+\frac{1}{2}$ stochasticity sets in through the interaction of the $1/n$ and $1/(n+1)$ resonances and $n^*\to n$ as $\xi\to 0$, equation (6) can as well be used in the $\xi=0$, $\nu=n+\frac{1}{2}$ case taking its limit for $\xi\to 0$. Let us now for the moment assume that (6) is valid for all values of ξ and ν^*. Then we can distinguish five different cases: the perpendicular ($\xi=0$), the quasi-perpendicular ($\xi^2/n^{*2}<(\alpha/r_0^2)|J''_{n^*}(r_0)|$), the oblique ($\xi^2/n^{*2}\simeq(\alpha/r_0^2)|J''_{n^*}(r_0)|$), the quasiparallel ($\xi^2/n^{*2}>(\alpha/r_0^2)|J''_{n^*}(r_0)|$) and the parallel ($1/\xi=0$). In the remaining part of this section we examine the form of the criterion (6) in each of these cases and compare it with numerical results.

(*a*) *Perpendicular case*

Taking the limit $\xi\to 0$ of (6) we find

$$f(r_0)=(|J_n(r_0)J''_n(r_0)|)^{1/2}/r_0\geqslant 1/4n\alpha. \tag{7}$$

Since for $r\gg n$ we have $J_n(r)\simeq -J''_n(r)$, equation (7) can be simplified to read

$$\tilde{f}(r_0)=|J_n(r_0)|/r_0\geqslant 1/4n\alpha \tag{8}$$

which is the criterion given by Fukuyama *et al* (1977) and Hsu (1982). The perpendicular case has been extensively discussed so that here we limit ourselves to a few points. First, by using the asymptotic formula $J_n(r)\simeq(2/\pi r)^{1/2}\cos(r-\pi n/2-\pi/4)$ in (8), we find

$$\Delta\alpha\simeq\tfrac{3}{8}(\tfrac{1}{2}\pi r_0/n)^{1/2}\,\Delta r=3\pi^{3/2}r_0^{1/2}/8\sqrt{2n}$$

(because in the region $r\gg\nu$ consecutive extrema of $J_n(r)$ are spaced at π apart) from which we see that as α is increased by $\Delta\alpha$ a new ring of chaotic trajectories is added in the xp_x surface of section, identical—up to a scaling factor—to the previously outermost ring and at a distance $\Delta r=\pi$ from it. Second, we see that the unusual properties of the perpendicular case, that attracted so much attention in this dynamical system, are due to the degeneracy of (3) and the non-monotonic form of the rotation number R (Karney 1978). Finally we note that asymptotic expansions of (7) and (8) around $r\simeq\nu$ do not give the same estimate for the 'stochasticity' threshold of the wave. From (7) we find

$$\alpha_{\text{thr}}\simeq(n^{1/3}/0.6754)[(n^2+1.6n^{4/3})/(1.6)n^{4/3}]^{1/2}\simeq 0.29n^{2/3} \tag{9a}$$

while from (8) we find

$$\alpha_{\text{thr}}\simeq(n+0.8n^{1/3})n^{1/3}/2.7n\simeq 0.37n^{1/3}. \tag{9b}$$

This clears the contradiction between the result $\alpha_{\text{thr}}\sim\nu^{1/3}$ (Fukuyama *et al* 1977) and $\alpha_{\text{thr}}\sim\nu^{2/3}$ (Karney 1978) that was pointed out by Hsu (1982). Of course neither of the two estimates fits accurately to the numerical results; (9*a*) overestimates α_{thr}—as is usually the case with the resonance overlap criterion—while (9*b*) underestimates α_{thr} due to the considerable difference in the values of the functions f and $\tilde{f}$ at $r\simeq\nu$.

(*b*) *Quasi-perpendicular case*

We use the term quasi-perpendicular for the case $\xi^2/n^{*2}<(\alpha/r_0^2)|J''_{n^*}(r_0)|$ not only because $\varphi\leqslant\pi/2$ in this case but mainly because the properties of such a system are found numerically to be very similar to a $\xi=0$ system. This becomes obvious if we neglect in (6) the term ξ^2/n^{*2} compared with $(\alpha/r_0^2)J''_{n^*}(r_0)$, in which case (6) becomes identical to (8) except for the dependence on ν^* and n^* instead of ν and n. One of the targets of this work is to define the range of φ for which the above inequality is satisfied, and we do it by looking at two extreme regimes, namely $r \simeq n^*$ and $r \gg n^*$, where simplifications by asymptotic formulae can yield straightforward answers. For $r \simeq n^*$ the condition $\xi/n^{*2} < (\alpha/r_0^2)|J''_{n^*}(r_0)|$ gives

$$\xi^2 < 0.67\alpha n^{*-1/3}, \qquad \alpha \geqslant \alpha_{\text{thr}}. \tag{10}$$

Taking $\alpha = \alpha_{\text{thr}} \simeq 0.37 n^{*1/3}$ in (10) we find an upper bound for ξ

$$\xi^2 < 0.25$$

that gives $\varphi > 63.5°$, in complete agreement with the result of Lichtenberg (1979). For $r \gg n^*$ one could expect that the r^{-2} factor in the $(\alpha/r^2)J''_{n^*}(r)$ term may impose a more severe restriction in ξ (and φ), but this is not the case. Using the asymptotic form for $J''_{n^*}(r)$ we find

$$\xi^2 < 0.115(n^*/\alpha^2)^{1/3}. \tag{11}$$

For $\alpha = \alpha_{\text{thr}}$ equation (11) becomes insensitive to variations in n^* and gives approximately $\varphi > 60°$.

From the above short discussion it is evident that the functional form of the criterion (9) and the general results of the cited references for perpendicular systems are still qualitatively valid for a range of propagation angles around $\pi/2$ as large as $\pi/2 \pm \pi/6$. In other words the characteristic zone-like picture of the surface of section of the perpendicular case (e.g. Karney 1978) persists in the quasi-perpendicular one. It should be emphasised though that the value of ν^* (and thus the location and the order of the island families) is not constant, but really depends on the evolution of the system through the relation

$$\nu^* = \nu + \xi^2 I_2. \tag{12}$$

An approximate relation between ν^* and r, useful in many applications, can be found in the following way. From the already known properties of a perpendicular system it is obvious that α, ν and r scale as $\alpha \leqslant \nu$ and $r > \nu$. Consequently we may neglect the wave term from the Hamiltonian (2), in which case (2) solved for I_2 gives

$$I_2 = [-\nu \pm (\nu^2 - 2\xi^2(I_1 - h))^{1/2}]/\xi^2 \tag{13}$$

where h is the (constant) value of the function H. Substituting I_2 from (13) into (12) we find

$$\nu^{*2} = \nu^2 - \xi^2(r^2 - 2h). \tag{14a}$$

Equation (14a) can be further simplified if, assuming $|p_z| \ll 2\nu^*/\xi$ (which is a reasonable condition in view of the fact that $\xi < 0.5$), we neglect from it the term $2h$. In this case ν^* is given by the simple relation

$$\nu^{*2} = \nu^2 - \xi^2 r^2. \tag{14b}$$

By replacing ν with ν^* from (14b) we can directly apply to quasi-perpendicular systems formulae already derived for perpendicular ones. For example equation (5) of Karney (1978), that gives the outer radius of the chaotic annulus, becomes

$$r_{\max} \simeq [(32/\pi)(\nu^2 - \xi^2 r_{\max}^2)\alpha^2]^{1/3}, \qquad \alpha > \alpha_{\text{thr}}. \tag{15}$$

Because in the $\alpha \gg \alpha_{\text{thr}}$ regime only the value of $r_{\max}$ is expected to be sensitive to changes of ξ, in table 1 we compare the values of $r_{\max}$ found numerically by Singh *et*

Table 1. Comparison between the numerical results of Singh *et al* (1981) and analytical estimates.

Method of computation	System I $\nu = 19.2$, $\alpha = 18.5$	System II $\nu = 4.8$, $\alpha = 4.6$
Numerically (from Singh *et al*)	88.5	22.8
$\xi = 0$ approximation (equation (7) graphically or (40) of Karney (1978) analytically)	110.0	22.0
$\xi = 0.167$ ($\varphi = 80.5°$) (equation (6) graphically or (15) analytically)	84.5	22.5

al (1981) with corresponding analytic estimates from equation (15) of this paper and (40) of Karney (1978). As we see the analytic estimates are in agreement with the computer experiment.

(*c*) *The oblique, quasi-parallel, and parallel cases*

An oblique dynamical system is defined here as a system of the form of (2) with $\xi = \cot^{-1}\varphi$ satisfying the relation

$$\xi / n^{*2} \approx (\alpha / r_0^2) |J''_{n^*}(r_0)|.$$

From the preceding discussion it is apparent that this implies almost always $\varphi < 60°$. In this case the derivative of the Bessel function in (7) can be neglected compared with the term ξ^2/n^{*2} and (6) becomes

$$|J_{n^*}(r_0)| \gtrsim 1/16\alpha\xi^2, \qquad \xi > 0.5. \tag{16}$$

This result (without the restriction in the angle) has been obtained by Smith and Kaufman (1978) (although in a slightly different form) by direct use of the two primary resonance overlap criterion.

An immediate result of (16) is the 'stochasticity' threshold of the wave amplitude

$$\alpha_{\text{thr}} \simeq 0.093 n^{*1/3}/\xi^2, \qquad \xi > 0.5, \tag{17}$$

which is always lower than the corresponding value given by (9*b*). Notice that (17) and (9) give the same α_{thr} for $\xi = 0.5$, which is consistent with the fact that α_{thr}, as defined in (9*b*) and (17), should depend continuously on ξ (and φ). However, the general properties of an oblique system are more involved than the corresponding properties of a quasi-perpendicular, since the former not only is genuinely two-dimensional, but has drifting rather than fixed resonances as well ($n^* = n^*(I_2)$). For instance, a direct estimate of the chaotic region from (16) is not possible since the rotation number R is in this case a monotonic function of r. Therefore (16) has meaning only around the first few (if not the first) maxima of $|J_{n^*}(r)|$. Consequently the main mechanism of chaotic trajectory 'diffusion' in this case is not the motion of the particle across the fixed $1/n$ (and possibly the $1/(n+1)$ for $\nu \simeq n + 0.5$) resonances (as it is in the perpendicular case), but rather the motion due to the drifting of ν^*, that in turn moves the first maximum of $J_{n^*}(r)$ ($J'_{n^*,1} \simeq n^* + 0.8n^{*1/3}$). An appropriate theory for oblique systems therefore should include the evolution of both harmonic oscillators $I_1\theta_1$ and $I_2\theta_2$ in time, using both surfaces of section. Moreover, this theory should in the limit $\xi^{-1} \to 0$ give the exactly parallel ($\xi^{-1} = 0$) case. The latter can be described by the separable Hamiltonian

$$H = H_1 + H_2,$$

$$H_1 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}(p_y - x)^2, \qquad H_2 = \tfrac{1}{2}p_z^2 - \alpha \sin(z - \nu t) = \tfrac{1}{2}I_2^2 - \nu I_2 - \alpha \sin\theta_2,$$

which is integrable, and has therefore no chaotic trajectories. We are presently looking in this direction, in order to treat the study of the oblique and quasi-parallel cases as a continuation of the perpendicular and quasi-perpendicular ones along the lines already discussed, in order to give a unified description of the chaotic regions on the xp_x and zp_z surfaces of section for all values of ξ from 0 to ∞. The results will be reported in a future publication.

4. Summary and discussion

In this work we discussed the large scale chaotic motion of a charged test particle in a static homogeneous magnetic field and a longitudinal electrostatic wave propagating at an angle φ with respect to the field. The motion of the particle is given by the Hamiltonian function (2) which describes two oscillators coupled through the wave term. In equation (2) ν is the ratio of the wave to the particle gyrofrequency, α is the dimensionless wave amplitude, and $\xi = \cot^{-1}\varphi$. We proposed a criterion for the onset of chaotic motion in this wave–particle system, and according to this criterion we divided the dynamical systems of the type of (2) into the following five classes.

(a) Perpendicular ($\varphi = 90°$). This is the case with the most clear-cut and unique characteristics, due to the degeneracy of its Hamiltonian, the most studied and the best understood. With α increasing from zero, trajectories are initially destabilised at a distance $r \simeq \nu + 0.8\nu^{1/3}$ from the origin of the surface of section, creating a toroidal shell of chaos in the (extended) phase space; on the surface of section xp_x this shell is represented by an annulus. As α further increases this chaotic annulus begins to spread, mainly outwards, in a characteristic two-step process. In step one a $1/n$ island family is successively created at a radial distance $\Delta r \simeq \pi$ from the last outermost $1/n$ family and rotated by an angle $\Delta\theta_1 = \pi/n$ to it. In step two this family is destabilised, either by the $1/(n+1)$ resonance (for $\nu \simeq n + 0.5$) or by a higher-order resonance (for all other values of ν), creating a new chaotic annulus. This process is described by (7) and its simplified version (8). In passing we note that the chaotic wandering of the particle in phase and velocity space can be described by a simple diffusion equation (e.g. Karney 1979, Antonsen and Ott 1981).

(b) Quasi-perpendicular ($90° > \varphi \gtrsim 65°$). As implied by its name this class has the general features of the perpendicular class but somehow 'smeared out'. The formulae of the perpendicular case can be corrected to take into account the finite ξ by using the Doppler shifted frequency ν^* from (12); this results in a shrinkage of the chaotic annulus of the corresponding perpendicular case.

(c) Oblique, quasi-parallel, parallel ($65° \gtrsim \varphi \geqslant 0°$). In the oblique case the rotation number becomes a monotonic function of r, which leads to a 'stochasticity' criterion with a stochasticity threshold lower than that of the corresponding perpendicular and quasi-perpendicular systems. Annuli with chaotic trajectories on the xp_x surface of section are created in this case as well, which are even narrower than in the corresponding quasi-perpendicular case, due to the strong dependence of ν^* on I_2. This was numerically observed by Singh *et al* (1983).

As $\varphi \to 0°$ the dynamical system begins to behave in a more orderly way, because the parallel ($\varphi = 0°$) case is integrable, thus possessing no chaotic trajectories at all. As a result the quasi-parallel case can be treated as a perturbation of the parallel, using the angle φ as the small parameter.

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Appendix 1

The Hamiltonian (2) can be transformed to a simpler form, in which resonances become explicit, by using a standard Lie transform algorithm to kill all resonances but two (Deprit 1969). In this algorithm the generating function W is calculated after a certain selection for the new Hamiltonian is made. In the present case we keep in the new Hamiltonian H^* all the angle independent and all the resonant terms. Taking $\nu^* = \nu + \xi^2 I_2$ and $\nu^* \simeq n^* + 0.5$ the new Hamiltonian is

$$\begin{aligned} H^* &= H_0^* + \alpha H_1^* + \tfrac{1}{2}\alpha^2 H_2^* + \ldots \\ &= I_1^* + \nu I_2^* + \tfrac{1}{2}\xi^2 I_2^{*2} - \alpha \sum_{l=n^*}^{n^*+1} J_l(r^*) \sin(l\theta_1^* - \theta_2^*) + \ldots \end{aligned} \tag{A1.1}$$

and the generating functions are

$$W = \alpha W_1 + \frac{\alpha^2}{2} W_2 + \ldots = \alpha \sum_{l \neq n^*, n^*+1} J_l(r^*) \frac{\cos(l\theta_1^* - \theta_2^*)}{l - \nu^*} + \frac{\alpha^2}{2} W_2 + \ldots$$

of the transformation $(\theta_1, \theta_2, I_1, I_2) \to (\theta_1^*, \theta_2^*, I_1^*, I_2^*)$ and

$$V = \alpha V_1 + \ldots = -\alpha W_1 + \ldots = -\alpha \sum_{l \neq n^*, n^*+1} J_l(r) \frac{\cos(l\theta_1 - \theta_2)}{l - \nu^*} + \ldots$$

of the inverse. The new action I_1^* is expressed as a function of the old variables as

$$I_1^* = I_1 + \frac{\partial V_1}{\partial \theta_1} + \ldots = I_1 + \alpha \sum_{l \neq n^*, n^*+1} \frac{l}{l - \nu^*} J_l(r) \sin(l\theta_1 - \theta_2) + \ldots. \tag{A1.2}$$

In the rest of this appendix we drop the stars from ν^* and n^* for notational convenience.

For the Hamiltonian (A1.1) to be used in place of (2), equation (A1.2) has to be a near identity transformation (that is, $I_1^* - I_1$ has to be of higher order than I_1) in order to use I_1 instead of I_1^* in H^*. To estimate the difference $I_1^* - I_1$ we first observe that

$$\sum_{l \neq n, n+1} \frac{l}{l - \nu} J_l(r) \sin(l\theta_1 - \theta_2) \leq \sum_{l \neq n, n+1} \frac{l}{l - \nu} J_l(r) = S. \tag{A1.3}$$

Noting that the main contribution to S comes from the terms with $l \simeq n$, we replace l

by n in the numerator of (A1.3) and we find

$$S \simeq -\frac{n}{1+\delta} J_{n-1}(r) - n \sum_{l=2}^{\infty} \left(\frac{J_{n-l}(r)}{\delta+l} + \frac{J_{n+l}(r)}{\delta-l} \right)$$

$$\simeq -nJ_{n-1}(r) - n \sum_{l=2}^{\infty} \frac{J_{n-l}(r) - J_{n+l}(r)}{l}$$

where $\delta = \nu - n \simeq 0.5$. The last sum can be approximately evaluated by using the asymptotic expansion of the Bessel functions, and by transforming the trigonometric sums to products. This gives

$$S \simeq -nJ_{n-1}(r) - 2n \left(\frac{2}{\pi r}\right)^{1/2} \sin\left(r - \frac{n\pi}{2} - \frac{\pi}{4}\right) \sum_{l=2}^{\infty} \frac{1}{l} \sin\frac{l\pi}{2}$$

$$\simeq -nJ_{n-1}(r) + 2nJ'_n(r) \sum_{l=1}^{\infty} \frac{(-1)^l}{2l+1}$$

$$= -nJ_{n-1}(r) + 2nJ'_n(r)(\pi/4 - 1)$$

and because $|J_l(r)| \leqslant 0.8 l^{-1/3}$ and $I_1 \geqslant \nu^2$ we conclude that $S \leqslant \mathrm{O}(n^{2/3})$ and $(I_1 - I_1^*)/I_1 \leqslant \mathrm{O}(\alpha \nu^{-4/3})$. The last result proves that (A1.2) is indeed a near identity transformation. Note that if we had kept only one resonant term in the new Hamiltonian (A1.1), $I_1 - I_1^*$ would be of lower order, and the restrictions for (A1.2) to be a near identity transformation would be severe (Karney 1978). Before closing we want to comment briefly on the form of H_2^* in (A1.1) and the second-order resonances of (2). Carrying on the algorithm to the next order we find that H_2^* can be split into two groups of terms, $H_2^* = H_{21}^* + H_{22}^*$. The first group is

$$H_{21}^* = \sum_{l \neq n^*, n^*+1} \frac{l}{l-\nu} \frac{\partial J^2(r^*)}{\partial I_1^*} \tag{A1.4}$$

and does not contribute to island formation. Note also that using equation (9.1.75) of Abramowitz and Stegun (1972) we can show that the sum in (A1.4) converges and that it is of second order with respect to H_1^*. The second group is angle dependent and more complicated, since it contains double sums. An estimate of its dominant behaviour (using the same equation above) gives

$$H_{22}^* \sim \sum_{m=2n^*}^{2n^*+2} J_m(2r^*) \cos(m\theta_1^* - 2\theta_2^*)$$

where the approximation is in the coefficients of the cosine terms. Obviously then the second-order resonances of (2) are the $2/2n^*$, $2/(2n^*+1)$, and $2/(2n^*+2)$.

Appendix 2

A Taylor expansion of (5) in the variable I_1 gives the approximate Hamiltonian

$$H_R \simeq H\,|_{I_{10}} + \tfrac{1}{2}\,\partial^2 H/\partial I_1^2\,|_{I_{10}} (\Delta I_1)^2. \tag{A2.1}$$

The width of an island of (A2.1) is given by (Ford 1978, Greene 1980).

$$\delta I_1 = 4\left(\frac{\alpha |J_{n^*}(r)|}{\xi^2/n^{*2} + \alpha^2 J''_{n^*}(r)/r^2}\right). \tag{A2.2}$$

On the other hand (5) implies that the onset of chaos is due to the interaction of the $R = 1/n^*$ and $R = (n^*+1)^{-1}$ resonances, where R (the rotation number of (5) along the radius passing from the centre of an n^*-island) is given by (Karney 1978)

$$R = (1/\nu^*)[1 + (\alpha/r)J'_{n^*}(r)]. \tag{A2.3}$$

From (A2.3) the distance ΔI_1 between the centre of an n^*- and an (n^*+1)-island is estimated as

$$\Delta I_1 \simeq \frac{\nu^*}{n^*(n^*+1)[\xi^2/n^{*2} + \alpha J''_{n^*}(r)/r^2]}. \tag{A2.4}$$

From (A2.2) and (A2.4) finally we find that the chaotic motion sets in when $\delta I_1 \geqslant \Delta I_1$, that is when

$$\{\alpha|J_{n^*}(r)|[\xi^2/n^{*2} + (\alpha/r^2)|J''_{n^*}(r)|]\}^{1/2} \geqslant \nu^*/4n^*(n^*+1) \simeq 1/4n^*. \tag{A2.5}$$

Note that the LHS of (A2.5) is evaluated at a local maximum (given approximately by $J'_{n^*}(r_0) = 0$) since the inequality is more easily satisfied at this point.

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