FIG. 4. Vertical and horizontal probes placed 10 cm apart in the plasma column.

nearest the horizontal probe shadow will also experience horizontal $\mathbf{E} \times \mathbf{B}$ and $\nabla p \times \mathbf{B}$ forces due to the electric field and density gradient at the edge of the horizontal probe shadow. Ion probe measurements show that the plasma has begun to fill in the space downstream from the horizontal probe by the time it reaches the vertical probe. Therefore, at the cross section of the vertical probe, the measured electric field and density gradient at the edge of the horizontal probe shadow are much less than the electric field and density gradient at the edge of the vertical probe shadow. Therefore, there was virtually no distortion of the vertical probe shadow as shown in Fig. 4.

There is no electric field or large density gradient at the central core of the plasma. The plasma core does not show any rotation, as indicated by the stationary structure of the core of the plasma column and by the lack of observable oscillations in this region. At the periphery of the column the $\mathbf{E} \times \mathbf{B}$ force is larger than the $\nabla p \times \mathbf{B}$ force; therefore, the plasma rotates in the $\mathbf{E} \times \mathbf{B}$ direction, and oscillations are observed. Producing an electric field and a pressure gradient at the center of the plasma column results in a dominant $\nabla p \times \mathbf{B}$ drift as shown in Fig. 4. This implies that producing an artificial boundary at regions of large electric fields may result in a pressure drift large enough to reduce the $\mathbf{E} \times \mathbf{B}$ mass motion.

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Comments

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Comments on "Enhanced Bremsstrahlung from Supraluminous, and Subluminous, Waves in Isotropic, Homogeneous Plasma"

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In a recent paper Lerche[1] presents a calculation of the enhanced transverse bremsstrahlung produced by two subluminous longitudinal waves from a plasma containing an isotropic ultrarelativistic tail. The calculation is similar to a nonrelativistic calculation of enhanced bremsstrahlung by Tidman and Dupree[2] which in turn is based on an emission formula derived by Dupree.[3] It is not obvious that by substituting relativistic spectral densities into these formulas one can find relativistically correct emission results. The relativistically correct emission formula was recently derived by Papadopoulos.[4] It is the purpose of this note to discuss the relationship of the results of Dupree and Papadopoulos, and to examine the accuracy of Lerche's[1] Eq. (2) for the situation he considered.

The emission formula derived by Papadopoulos[4] is (using the notation of Ref. 1)

$$\frac{dI}{d\omega} = \frac{|\omega|\,(\omega^2 - \omega_e^2)^{1/2}}{8\pi^5 c^3} \sum_{\alpha,\beta,\gamma,\delta} \frac{\omega_\alpha^2 \omega_\beta^2 q_\gamma q_\delta}{\omega^2} \int \frac{d\mathbf{K}'}{K'^2} \int d\omega' \cdot [(K')^{-2} T^{\alpha\beta}(\mathbf{K} - \mathbf{K}', \omega - \omega') S^{\gamma\delta}(\mathbf{K}', \omega') + |\mathbf{K} - \mathbf{K}'|^{-2}\, \Lambda^{\alpha\delta}(\mathbf{K}', \omega') \Lambda^{\beta\gamma}(\mathbf{K} - \mathbf{K}', \omega - \omega')]. \tag{1}$$

The following definitions are used:

$$T^{\alpha\beta}(\mathbf{K}', \omega') = \langle Q_{1\perp}^{\alpha} Q_{1\perp}^{\beta} \mid \mathbf{K}', \omega' \rangle, \tag{2}$$

$$\Lambda^{\alpha\beta}(\mathbf{K}', \omega') = \langle Q_{1\perp}^{\alpha} n_1^{\beta} \mid \mathbf{K}', \omega' \rangle, \tag{3}$$

$$\mathbf{Q}_1^{\alpha}(\mathbf{K}', \omega') = m_\alpha \int d\mathbf{p} f_1^{\alpha}(\mathbf{K}', \omega', \mathbf{p}) \mathbf{g}(\mathbf{K}'; \mathbf{K}, \omega; \mathbf{p}), \tag{4}$$

$$\mathbf{g}(\mathbf{K}'; \mathbf{K}, \omega; \mathbf{p}) = \mathbf{K}' \cdot \frac{\partial}{\partial \mathbf{p}} [\mathbf{v}(1 - \mathbf{K} \cdot \mathbf{v}\omega^{-1})^{-1}]. \tag{5}$$

Here $f_1^{\alpha}(\mathbf{K}', \omega', \mathbf{p})$ is the usual first-order perturbation to the distribution function and subscript $\perp$ means "transverse to the emitted wave vector $\mathbf{K}$."

Since our primary interest is the emission from electron–electron encounters, we restrict our analysis to this case. This simplifies the mathematics without affecting the generality of the conclusions.[5] In this case, Eq. (1) can be cast in the form

$$\frac{dI^{ee}}{d\omega} = \frac{|\omega|\,(\omega^2 - \omega_e^2)^{1/2}\omega_e^4 e^2}{8\pi^5 c^3} \int \frac{d\mathbf{K}'}{K'^2} \int d\omega'$$
$$\cdot [(K')^{-2} T^{ee}(\mathbf{K} - \mathbf{K}', \omega - \omega') S^{ee}(\mathbf{K}', \omega')$$
$$+ |\mathbf{K} - \mathbf{K}'|^{-2}\, \Lambda^{ee}(\mathbf{K} - \mathbf{K}', \omega - \omega') \Lambda^{ee}(\mathbf{K}', \omega')]. \quad (6)$$

The correlation functions S^{ee}, T^{ee}, and Λ^{ee} can be calculated using Rostoker's test-particle model.[6] If we consider the ions as forming a uniform neutralizing background, then Eqs. (2)–(4) become

$$S^{ee}(\mathbf{K}', \omega') = |D(\mathbf{K}', \omega')|^{-2}\, \langle n_0^e n_0^e \mid \mathbf{K}', \omega' \rangle, \quad (7)$$

$$T_e^{ee}(\mathbf{K}', \omega')$$
$$= |D(\mathbf{K}', \omega')|^{-2}\, [|1 - L_e(\mathbf{K}', \omega')|^2 \langle Q_0^e Q_0^e \mid \mathbf{K}', \omega' \rangle$$
$$+ 2(\mathrm{Re}\, \Delta^e - \mathrm{Re}\, L_e\, \mathrm{Re}\, \Delta^e - \mathrm{Im}\, L_e\, \mathrm{Im}\, \Delta^e)$$
$$\cdot \langle Q_0^e n_0^e \mid \mathbf{K}', \omega' \rangle$$
$$+ |\Delta^e(\mathbf{K}', \omega')|^2 \langle n_0^e n_0^e \mid \mathbf{K}', \omega' \rangle]; \quad (8)$$

$$\Lambda^{ee}(\mathbf{K}', \omega')$$
$$= |D(\mathbf{K}', \omega')|^{-2}\, (|1 - \mathrm{Re}\, L_e|\, \langle Q_0^e n_0^e \mid \mathbf{K}', \omega'$$
$$+ \mathrm{Re}\, \Delta^e \langle n_0^e n_0^e \mid \mathbf{K}', \omega' \rangle), \quad (9)$$

where the dielectric properties of the plasma are given by

$$[L_e(\mathbf{K}', \omega'); \Delta^e(\mathbf{K}', \omega')] = \omega_e^2 m_e (\mathbf{K}')^{-2}$$
$$\cdot \int d\mathbf{p}\, \mathbf{K}' \cdot \frac{\partial f_0^e}{\partial \mathbf{p}} \cdot (\mathbf{K}' \cdot \mathbf{v} - \omega')[1; g_\perp(\mathbf{K}'; \mathbf{K}, \omega; \mathbf{p})], \quad (10)$$

$$D(\mathbf{K}', \omega') = 1 - L_e(\mathbf{K}', \omega'). \quad (11)$$

Also the correlation functions of the particles moving in their unperturbed trajectories are expressed by

$$\begin{bmatrix} \langle n_0^e n_0^e \mid \mathbf{K}', \omega' \rangle \\ \langle n_0^e Q_0^e \mid \mathbf{K}', \omega' \rangle \\ \langle Q_0^e Q_0^e \mid \mathbf{K}', \omega' \rangle \end{bmatrix} = 2\pi \int d\mathbf{p}\; \delta(\omega' - \mathbf{K}' \cdot \mathbf{v}) f_0^e(\mathbf{p})$$
$$\cdot \begin{bmatrix} 1 \\ m_e g_\perp(\mathbf{K}'; \mathbf{K}, \omega; \mathbf{p}) \\ m_e^2 g_\perp^2(\mathbf{K}'; \mathbf{K}, \omega; \mathbf{p}) \end{bmatrix}. \quad (12)$$

Equations (6)–(12) accurately describe the electron–electron emission from a plasma in terms of the electron distribution function. The plasma may be relativistic or not.

We now notice that if we approximate

$$\mathbf{K}' \cdot (\partial/\partial \mathbf{p}) \{\mathbf{v}_\perp [1 - (\mathbf{K} \cdot \mathbf{v}/\omega)]^{-1}\} \text{ by } \mathbf{K}_\perp' m_e^{-1},$$

then

$$\Delta^e \to m_e^{-1} K_\perp' L_e,$$

$$\langle Q_0^e Q_0^e \mid \mathbf{K}', \omega' \rangle \to m_e^{-2} K_\perp'^2 \langle n_0^e n_0^e \mid \mathbf{K}', \omega' \rangle,$$

and

$$\langle Q_0^e n_0^e \mid \mathbf{K}', \omega' \rangle \to m_e^{-1} K_\perp' \langle n_0^e n_0^e \mid \mathbf{K}', \omega' \rangle.$$

Using the above approximation Dupree's formula is recovered. The effects of such an assumption and the resulting error depend on the particular situation under discussion. However, some general remarks can be made without reference to particular situations. If the relevant velocities are relativistic, the substitution $\partial/\partial \mathbf{p} \to m_e^{-1}(\partial/\partial \mathbf{v})$ will result in a difference of the order of the Jacobian, $J(|\mathbf{v}|/|\mathbf{p}|)$, which can be substantial. For generation of transverse waves with phase velocity $\omega > cK$ but $\omega \simeq cK$, the approximation $1 - \mathbf{K} \cdot \mathbf{v}/\omega \sim 1$ may be rather poor and some care must be exercised with regard to the number of terms that must be kept in expanding $[1 - (\mathbf{K} \cdot \mathbf{v}/\omega)]^{-1}$ as a power series in $(\mathbf{K} \cdot \mathbf{v}/\omega)$. Dupree's formula has been used[2] to calculate quadrupole electron–electron radiation (the dipole electron–electron radiation being zero). Although quadrupole radiation was found by expanding the density correlations with argument $(\mathbf{K} - \mathbf{K}')$ in a Taylor series with respect to $|\mathbf{K}| \cdot |\mathbf{K}'|^{-1}$, the final result underestimated the radiation because some terms of quadrupole order were omitted {e.g., $(\mathbf{K} \cdot \mathbf{v}/\omega)^2$ from the expansion of $[1 - (\mathbf{K} \cdot \mathbf{v}/\omega)]^{-1}$, and v^2/c^2 from $J(|\mathbf{v}|/|\mathbf{p}|)$}.

We now examine the particular situation considered by Lerche.[1] An electron plasma with distribution function

$$f_0^e(|\mathbf{p}|) = f_e \text{ (thermal)} + \frac{n_r}{n_T} f_r(|\mathbf{p}|) \quad (13)$$

with

$$f_r = \left(\frac{p}{p_0}\right)^{-\gamma} \frac{(\gamma - 1)}{4\pi p_1^2}; \qquad \gamma > 3, p > p_1$$

is immersed in a neutralizing ion background. Here, $f_r(|\mathbf{p}|)$ represents a relativistic tail and n_r, n_T are the number densities of the relativistic and thermal electrons, respectively, with $n_r n_T^{-1} \ll 1$. We are interested in the transverse emission at frequency $\sim 2\omega_e$, due to the "collision" of two longitudinal fluctuations at frequencies $\sim \omega_e$. Thus, we require the values of S^{ee}, T^{ee}, etc., at the resonances of $|D(\mathbf{K}', \omega'|^{-1}$. From Eqs. (6)–(12) we find that at resonance

$$\frac{dI^{ee}}{d\omega} \simeq \frac{|\omega|\,(\omega^2 - \omega_e^2)^{1/2}}{8\pi^5 c^3} \frac{\omega_e^4 e^2}{\omega^2} \frac{\pi^2 \omega_e^2}{4} \int \frac{d\mathbf{K}'}{K'^2}$$

$$\cdot \{ [|\mathrm{Re}\, \Delta^e(\mathbf{K} - \mathbf{K}', \omega_e)|^2 (K')^{-2}$$

$$+ |\mathbf{K} - \mathbf{K}'|^{-2}\, \mathrm{Re}\, \Delta^e(\mathbf{K} - \mathbf{K}', \omega_e)\, \mathrm{Re}\, \Delta^e(\mathbf{K}', \omega_e)]$$

$$+ \frac{\langle n_0^e n_0^e \mid \mathbf{K}', \omega_e \rangle \langle n_0^e n_0^e \mid \mathbf{K} - \mathbf{K}', \omega_e \rangle}{|\mathrm{Im}\, D(\mathbf{K} - \mathbf{K}', \omega_e)|\; |\mathrm{Im}\, D(\mathbf{K}', \omega_e)|}$$

$$\cdot [\delta(\omega - 2\omega_e) + \delta(\omega + 2\omega_e)] \} \tag{14}$$

since

$$|D(\mathbf{K}', \omega')|^{-1} \simeq \frac{\pi\omega_e[\delta(\omega' - \omega_e) + \delta(\omega' + \omega_e)]}{2\, |\mathrm{Im}\, D(\mathbf{K}', \omega_e')|}. \tag{15}$$

We are mainly concerned with waves which resonate with relativistic particles so that the domain of K' is[1] $\omega_e/c < K' < (\omega_e/v_{Th})\alpha^{-1}$. The balance of emission and absorption for these phase velocities is controlled by the relativistic particles. Thus, the dominant contribution to

$$\Phi_1(\mathbf{K}, \mathbf{K}') \equiv \frac{\langle n_0^e n_0^e \mid \mathbf{K}', \omega_e \rangle \langle n_0^e n_0^e \mid \mathbf{K} - \mathbf{K}', \omega_e \rangle}{|\mathrm{Im}\, D(\mathbf{K}', \omega_e)|\; |\mathrm{Im}\, D(\mathbf{K} - \mathbf{K}', \omega_e)|} \tag{16}$$

is given by the relativistic part of the distribution function.[1] For the function given by (13) we obtain

$$\Phi_1(\mathbf{K}, \mathbf{K}') = \frac{4K'^2\, |\mathbf{K} - \mathbf{K}'|^2 c^4 \epsilon_1^2}{\omega_e^6\, |\omega'|\, |\omega - \omega'|\, n_T^2 (\gamma - 1)^2}. \tag{17}$$

However, since $n_T v_{Th}^2 \gg n_r c^2$ the dominant part of $\Delta^e(\mathbf{K}', \omega_e)$ is due to the thermal electrons. Then, for a thermal Maxwellian

$$\Phi_2(\mathbf{K}, \mathbf{K}') \equiv |\mathrm{Re}\, \Delta^e(\mathbf{K} - \mathbf{K}', \omega_e)|^2 (\mathbf{K}')^{-2}$$

$$+ |\mathbf{K} - \mathbf{K}'|^{-2}\, \mathrm{Re}\, \Delta^e(\mathbf{K} - \mathbf{K}', \omega_e)\, \mathrm{Re}\, \Delta^e(\mathbf{K}', \omega_e)$$

$$\simeq 2\, \frac{(K^2 - 2\mathbf{K}\cdot\mathbf{K}')(K'_\perp)^2}{K'^2\, |\mathbf{K} - \mathbf{K}'|^2}. \tag{18}$$

Combining (14)–(18) we obtain

$$I^{ee}(2\omega_e) \simeq \frac{2\sqrt{3}\,\epsilon_1^2}{3\pi^2(\gamma - 1)^2\alpha^3} \left(\frac{e^2\omega_e^5}{c^3 v_{th}}\right)\left(\frac{c^2}{v_{th}^2}\right). \tag{19}$$

Comparing (19) with Eq. (61) of Ref. 1 we find a numerical factor of 15 larger. However, the functional dependence of the two results is identical. The reason for the same functional dependence is the existence in both cases of the factor $\Phi_1(\mathbf{K}, \mathbf{K}')$. In the cases considered in Ref. (1) and here, it is this factor which controls the balance of waves. This factor appears in exactly the same form in the nonrelativistic and the relativistic treatment, although its specific evaluation depends on the form of the distribution function. The factor 15 comes from $\Phi_2(\mathbf{K}, \mathbf{K}')$ which is different in the relativistic and nonrelativistic cases since K' can approach ω_e/c from above.

The final intensity of emission is extremely sensitive to the choice of distribution function for the ultrarelativistic electrons, e.g., a further enhancement by about a factor of 10 appears in Eq. (18) for the case of relativistic Maxwellian tail, of the same mean energy per particle as the $p^{-\gamma}$ distribution. In view of this and the uncertainties involved in the parameters for astrophysical plasmas we conclude that the results given by Lerche adequately represent the situation which he investigated.

However, in the more general context of bremsstrahlung from plasmas which contain or are suspected of containing a relativistic, or even mildly relativistic, component it is clear that the general formula Eq. (6) should be used. The specific results of either Lerche or Papadopoulos, who evaluated the radiation formulas for choices of ultrarelativistic and thermal plasma distribution functions [viz., $p^{-\alpha}$ for Lerche; exp $(-\sigma|\mathbf{p}|)$ for Papadopoulos] are valid only within the framework of first approximations to Eq. (6).

We believe that the general functional dependence of the enhanced bremsstrahlung at $\sim 2\omega_e$ will remain the same for nearly all physical choices of thermal and relativistic distribution functions. Of course it is always possible to choose a distribution function which disproves the above statement, however the "cooking conditions" for such a distribution function would themselves have to be particular rather than general.

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[5] The case of electron–ion collisions would go through in a similar manner. We have omitted this case in the interests of brevity and clarity of discussion.

[6] In Papadopoulos' calculation the test-particle fields only included *subluminous* electrostatic components. The supraluminous electrostatic components were omitted. In these comments we shall not discuss the controversial question of the existence of these supraluminous waves.