Radiation from a localized Langmuir oscillation in a uniformly magnetized plasma

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The electromagnetic radiation at the first and second harmonics of the electron plasma frequency from a localized Langmuir perturbation is computed for the case of a uniformly magnetized plasma. It is assumed that the localized perturbations in both the electrostatic field and plasma density have cylindrical symmetry about the direction of the ambient magnetic field. The analysis is initially performed for such an arbitrary localized perturbation, and then applied to treat the case of a quasi-planar Langmuir soliton propagating in the direction of the magnetic field. An extensive numerical study of the angular dependence of the radiation spectrum as a function of the ratio of the electron plasma and cyclotron frequencies is described.

I. INTRODUCTION

The question of strong Langmuir turbulence in magnetized plasmas is important in studies of beam-plasma interactions in both space and laboratory plasmas. Heretofore, studies of strong turbulence theory in magnetized plasmas have centered on the dynamics of collapse and the shape and stability of the localized structures which result. However, such structures are expected to have electromagnetic signatures at harmonics of the electron plasma frequency. While the electromagnetic radiation from Langmuir solitons has been extensively studied for field-free plasmas,[1-5] there has been scant treatment of the problem of the strongly turbulent radiation process in magnetized plasmas. This problem is of particular relevance due to increased interest in experimental studies of electron beam driven strong turbulence in the laboratory.[6-10] It is our intention to address this question in the present work, and to derive expressions for the radiation emissivity from spiky Langmuir turbulence at the first and second harmonics of the electron plasma frequency.

The organization of the paper is as follows: In Sec. II, we derive an expression for the emissivity from an arbitrary, cylindrically symmetric soliton at frequencies $\omega \simeq \omega_e$, $2\omega_e$ (where ω_e denotes the electron plasma frequency). It should be noted that the treatment of emission at the electron plasma frequency is restricted to the limit in which the radiation wavelength is much less than the scale length of the soliton. In order to investigate simplified scaling laws between the radiation emissivity and the soliton amplitude, we consider the limiting case of one-dimensional Langmuir solitons in Sec. III. A numerical study of the angular dependence of the emissivity is also presented in this section. In particular, we investigate the variation of the radiation pattern with ω_e/Ω_e (where Ω_e is the electron cyclotron frequency). A summary and discussion appears in Sec. IV, and the derivation of the plasma dispersion tensor and the radiation source current is given in the Appendix.

II. THE EMISSIVITY

We assume a localized Langmuir perturbation of the form

$$\mathbf{E}(\mathbf{x},t) = \nabla\phi(r,z)\sin\omega_e t\,, \tag{1}$$

where $\nabla\phi(r,z)$ defines the soliton envelope, and the ambient magnetic field $\mathbf{B}_0(\equiv B_0\hat{\mathbf{e}}_z)$ defines the z axis. The interaction between the electrostatic field and the associated slow time scale oscillation in the plasma density is implicitly included, and we use $\delta n(r,z)$ to denote the density caviton. Both $\phi(r,z)$ and $\delta n(r,z)$ possess cylindrical symmetry about the z axis, and it is assumed that $\phi(r,z)$ and $\delta n(r,z)$ are odd and even functions of z, respectively.

The radiated power is defined to be

$$P = -\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt \int d^3x\, \delta\mathbf{E}(\mathbf{x},t)\cdot\delta\mathbf{J}_s(\mathbf{x},t)\,, \tag{2}$$

where $\delta\mathbf{E}(\mathbf{x},t)$ is the radiation electric field, $\delta\mathbf{J}_s(\mathbf{x},t)$ is the source current due to the localized Langmuir perturbation, and the spatial intergration is over all space. We implicitly assume, in this analysis, that a steady state has been achieved in which the soliton structure changes, at worst, slowly in time. Such a state may be formed, for example, due to the nonlinear stabilization of electron beam-plasma instabilities,[11] where the linear growth of a resonant spectrum is balanced by the nonlinear transfer of wave energy to modes not in resonance with the beam. Equation (2) can be expressed in terms of the Fourier amplitudes of $\delta\mathbf{E}(\mathbf{x},t)$ and $\delta\mathbf{J}_s(\mathbf{x},t)$ in the following manner:

$$P = -(2\pi)^4\lim_{T\to\infty}\frac{1}{T}\int d^3k \int_{-\infty}^{\infty} d\omega\, \delta\mathbf{E}(\mathbf{k},\omega)\cdot\delta\mathbf{J}_s^*(\mathbf{k},\omega)\,, \tag{3}$$

where the asterisk (*) denotes the complex conjugate, and the Fourier transform is defined as

$$\mathbf{f}(\mathbf{k},\omega) = (2\pi)^{-4}\int d^3x \int_{-\infty}^{\infty} dt\, \exp(i\omega t - i\mathbf{k}\cdot\mathbf{x})\mathbf{f}(\mathbf{x},t)\,. \tag{4}$$

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A self-consistent relation between the radiation field and source current is derived in the Appendix and is of the form

$$\mathbf{\Lambda}(\mathbf{k},\omega)\cdot\delta\mathbf{E}(\mathbf{k},\omega)=-(4\pi i/\omega)\delta\mathbf{J}_s(\mathbf{k},\omega), \tag{5}$$

where the dispersion tensor is given by $(\mathbf{k}=k_x\hat{\mathbf{e}}_x+k_z\hat{\mathbf{e}}_z)$

$$\mathbf{\Lambda}(\mathbf{k},\omega)=(c^2/\omega^2)(\mathbf{kk}-k^2\mathsf{I})+\boldsymbol{\epsilon}(\mathbf{k},\omega). \tag{6}$$

In Eq. (6), I is the unit dyadic, $\boldsymbol{\epsilon}(k,\omega)$ is the plasma dielectric tensor, and we have that

$$\epsilon_{xx}=\epsilon_{yy}=\epsilon_1\,,\quad \epsilon_{xy}=-\epsilon_{yx}=i\epsilon_2\,,\quad \epsilon_{zz}=\epsilon_3\,,$$

$$\epsilon_{xz}=\epsilon_{zx}=\epsilon_{yz}=\epsilon_{zy}=0\,,$$

where $\epsilon_1\equiv 1-\omega_e'^2/(\omega^2-\Omega_e^2)$, $\epsilon_2\equiv\omega_e'^2\Omega_e/\omega(\omega^2-\Omega_e^2)$, and $\epsilon_3\equiv 1-\omega_e'^2/\omega^2$. Here, $\omega_e'=\omega_e(1+\delta n_{ex}/n_0)^{1/2}$, where n_0 is the ambient electron density and δn_{ex} denotes the extremum of δn. This dispersion tensor includes the nonlinear modifications to the cold plasma approximation and has been derived under the restriction that the wavelength of the radiation be much less than the scale length of the local perturbation.

Inverting Eq. (5) to find $\delta\mathbf{E}(\mathbf{k},\omega)$ as a function of $\delta\mathbf{J}_s(\mathbf{k},\omega)$, we obtain[12]

$$\delta\mathbf{E}(\mathbf{k},\omega)=-(4\pi i/\omega)[\lambda_{ss}(\mathbf{k},\omega)/\Lambda(\mathbf{k},\omega)]\times\hat{\mathbf{a}}(\mathbf{k},\omega)\hat{\mathbf{a}}^*(\mathbf{k},\omega)\cdot\delta\mathbf{J}_s(\mathbf{k},\omega)\,, \tag{7}$$

where $\Lambda(\mathbf{k},\omega)$ is the determinant of $\mathbf{\Lambda}(\mathbf{k},\omega)$, $\lambda_{ss}(\mathbf{k},\omega)$ is the trace of the classical adjoint of $\mathbf{\Lambda}(\mathbf{k},\omega)$, and $\hat{\mathbf{a}}(\mathbf{k},\omega)=\delta\mathbf{E}(\mathbf{k},\omega)/|\delta\mathbf{E}(\mathbf{k},\omega)|$ is the unit polarization vector.

The power radiated per unit solid angle subtended by $\mathbf{k}$ is found by substitution of Eq. (7) into (3), and it can be shown that

$$\frac{dP}{d\Omega_{\mathbf{k}}}=(2\pi)^6\lim_{T\to\infty}\frac{1}{T}\sum_{\pm}\int_0^\infty dk^2k\int_0^\infty\frac{d\omega}{\omega}\frac{\lambda_{ss}(\mathbf{k},\omega)}{|(\partial/\partial k^2)\Lambda(\mathbf{k},\omega)|_{k^2=k_\pm^2}}\times|\hat{\mathbf{a}}_{(\pm)}(\mathbf{k},\omega)\cdot\delta\mathbf{J}_s^*(\mathbf{k},\omega)|^2\delta(k^2-k_\pm^2)\,, \tag{8}$$

where $\sum_\pm$ denotes a sum over the wave modes of the system, $d\Omega_{\mathbf{k}}=2\pi\sin\theta\,d\theta$ (where θ denotes the angle between $\mathbf{k}$ and $\mathbf{B}_0$), and we have summed over the contributions of positive and negative ω. The appropriate wave modes are described by

$$\frac{c^2k_\pm^2}{\omega^2}=1-\frac{2\alpha^2(1-\alpha^2)}{2(1-\alpha^2)-\beta^2(\sin^2\theta\mp\rho)}\,, \tag{9}$$

where the "+" and "−" denote the ordinary and extraordinary modes, respectively, $\alpha^2\equiv\omega_e'^2/\omega^2$, $\beta^2\equiv\Omega_e^2/\omega^2$, and

$$\rho^2=\sin^4\theta+4(\omega/\Omega_e)^2(1-\alpha^2)^2\cos^2\theta\,.$$

This corresponds to the well-known Appleton–Hartree dispersion relation in which ω_e' has been substituted for ω_e to describe the nonlinear effect.

In this paper we treat emission at $\omega\simeq\omega_e$, $2\omega_e$ and write the source current as the sum of contributions (see the Appendix)

$$\delta\mathbf{J}_s(\mathbf{k},\omega)=\delta\mathbf{J}_s^{(1)}(\mathbf{k})\delta(\omega-\omega_e)+\delta\mathbf{J}_s^{(2)}(\mathbf{k})\delta(\omega-2\omega_e)\,,$$

where

$$\delta\mathbf{J}_s^{(1)}(\mathbf{k})=-(2\pi)^{-4}\frac{\omega_e}{4}\int d^3x\exp(-i\mathbf{k}\cdot\mathbf{x})\frac{\delta n}{n_0}\left(\nabla\phi+\frac{\Omega_e^2}{\omega_e^2-\Omega_e^2}\nabla_\perp\phi+i\frac{\omega_e\Omega_e}{\omega_e^2-\Omega_e^2}\hat{\mathbf{e}}_z\times\nabla\phi\right), \tag{10}$$

$$\begin{aligned}\delta\mathbf{J}_s^{(2)}(\mathbf{k})=i(2\pi)^{-4}\frac{e}{8m_e\omega_e}\int d^3x\exp(-i\mathbf{k}\cdot\mathbf{x})\Bigg[&\left(\nabla^2\phi+\frac{\Omega_e^2}{\omega_e^2-\Omega_e^2}\nabla_\perp^2\phi\right)\left(\nabla\phi+\frac{\Omega_e^2}{\omega_e^2-\Omega_e^2}\nabla_\perp\phi+i\frac{\omega_e\Omega_e^2}{\omega_e^2-\Omega_e^2}[3\mathsf{I}-2\boldsymbol{\epsilon}^*(\mathbf{k},2\omega_e)]\cdot(\hat{\mathbf{e}}_z\times\nabla\phi)\right)\\&+[\mathsf{I}-\boldsymbol{\epsilon}^*(\mathbf{k},2\omega_e)]\cdot\left(\nabla+\frac{\Omega_e^2}{\omega_e^2-\Omega_e^2}\nabla_\perp\right)\left((\nabla\phi)^2+\frac{\Omega_e^2}{\omega_e^2-\Omega_e^2}(\nabla_\perp\phi)^2\right)\Bigg]\,,\end{aligned} \tag{11}$$

and $\nabla_\perp\equiv\nabla-\hat{\mathbf{e}}_z(\partial/\partial z)$. If we write the square of the delta function as $\delta^2(\omega-\omega_0)=\lim_{T\to\infty}(T/2\pi)\delta(\omega-\omega_0)$ and evaluate $\partial\Lambda/\partial k^2$ for the appropriate mode, then the expression for the radiated power becomes

$$\frac{dP}{d\Omega_{\mathbf{k}}}=(2\pi)^5\sum_\pm\int_0^\infty dk^2k\int_0^\infty d\omega\frac{\omega^2\lambda_{ss}(\mathbf{k},\omega)}{c^2\Omega_e\rho\epsilon_2}\delta(k^2-k_\pm^2)\times[|\hat{\mathbf{a}}_{(\pm)}(\mathbf{k},\omega)\cdot\delta\mathbf{J}_s^{(1)*}(\mathbf{k})|^2\delta(\omega-\omega_e)+|\hat{\mathbf{a}}_\pm(\mathbf{k},\omega)\cdot\delta\mathbf{J}_s^{(2)}(\mathbf{k})|^2\delta(\omega-2\omega_e)]\,. \tag{12}$$

In the evaluation of the radiated power at $\omega\simeq\omega_e$ and $2\omega_e$ which follows, we make use of the expression[12]

$$\hat{\mathbf{a}}=-i(\lambda_{ss}\lambda_{22})^{1/2}(\lambda_{12},\lambda_{22},\lambda_{32})$$

to describe the unit polarization vector, where $\boldsymbol{\lambda}$ is the classical adjoint of $\mathbf{\Lambda}$. It should be noted, however, that this approach is invalid when the eigenvalues of $\mathbf{\Lambda}$ are degenerate (i.e., when $k_+^2=k_-^2$). Since this occurs in the limit in which $\Omega_e\to 0$, care must be exercised in order to treat the field-free case.

A. Emission at $\omega\simeq\omega_e$

In this frequency regime we are restricted to consideration of waves whose wavelength is much less than the scale length of the perturbation, which is equivalent, in practice, to the condition that $c^2k_\pm^2/\omega_e^2>|\delta n_{ex}/n_0|$. The emissivity $\eta[\equiv\omega^{-1}dP(\omega)/d\Omega_{\mathbf{k}}]$ is defined to be the power radiated per unit frequency per unit solid angle subtended by $\mathbf{k}$. In computing $\eta(\omega_e,\theta)$, we retain only the contribution due to the oscillatory current at ω_e [i.e., $\delta\mathbf{J}_s^{(1)}(\mathbf{k})$] and find

$$\eta(\omega_e,\theta)=\frac{\omega_e^3}{64\pi c^3}\left(\frac{\delta n_{ex}}{n_0}\right)^{-2}\sum_\pm\frac{N_\pm(\rho\pm\sin^2\theta)}{\rho\cos^2\theta}\times\left[N_\pm^2\sin\theta\cos\theta I_\parallel-\frac{\omega_e^2}{\omega_e^2-\Omega_e^2}\times\left(N_\pm^2\sin^2\theta+\frac{\delta n_{ex}}{n_0}-\frac{\Omega_e^2}{2\omega_e^2}(\sin^2\theta\mp\rho)\right)I_\perp\right]^2\,, \tag{13}$$

where

$$I_{\parallel} \equiv \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dr\, r \cos(k_{\perp} z \cos\theta) J_0(k_{\perp} r \sin\theta) \frac{\delta n}{n_0} \nabla_{\parallel}\phi\ , \tag{14}$$

$$I_{\perp} = \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dr\, r \sin(k_{\pm} z \cos\theta) J_1(k_{\pm} r \sin\theta) \frac{\delta n}{n_0} \nabla_{\perp}\phi\ , \tag{15}$$

$$\rho^2 = \sin^4\theta + 4(\omega_e^2/\Omega_e^2)(\delta n_{ex}/n_0)^2 \cos^2\theta\ ,$$

$N(\equiv ck/\omega_e)$ is the index of refraction,

$$N_{\pm}^2 = 1 - \frac{\delta n_{ex}}{n_0}\left(1 + \frac{\delta n_{ex}}{n_0}\right)\left(\frac{\delta n_{ex}}{n_0} + \frac{\Omega_e^2}{2\omega_e^2}(\sin^2\theta \pm \rho)\right)^{-1}, \tag{16}$$

and J_n are the regular Bessel functions of order n.

B. Emission at $\omega \simeq 2\omega_e$

In this frequency regime, the nonlinear contributions to the dielectric properties of the plasma can be ignored, and the Appleton–Hartree dispersion relation[13] can be employed. After retaining only the oscillatory source current at $2\omega_e$ in Eq. (12), we find that

$$\eta(2\omega_e, \theta) = \frac{e^2}{36\pi m_e^2 c^3 \omega_e} \sum_{\pm} \frac{N_{\pm}(\rho \pm \sin^2\theta)}{\rho \cos^2\theta} \times (A_{\parallel,\parallel} I_{\parallel,\parallel} + A_{\parallel,\perp} I_{\parallel,\perp} + A_{\perp,\parallel} I_{\perp,\parallel} + A_{\perp,\perp} I_{\perp,\perp})^2\ , \tag{17}$$

where

$$\begin{aligned} I_{\parallel,\parallel} &= \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dr\, r \cos(k_{\pm} z \cos\theta) \times J_0(k_{\pm} r \sin\theta)(\nabla\phi)_{\parallel}^2\ , \\ I_{\parallel,\perp} &= \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dr \sin(k_{\pm} z \cos\theta) \times J_0(k_{\pm} r \sin\theta)(\nabla\phi)_{\parallel}(\nabla\phi)_{\perp}\ , \\ I_{\perp,\parallel} &= \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dr\, r \sin(k_{\pm} z \cos\theta) \times J_1(k_{\pm} r \sin\theta)(\nabla\phi)_{\parallel}(\nabla\phi)_{\perp}\ , \\ I_{\perp,\perp} &= \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dr\, r \cos(k_{\pm} z \cos\theta) \times J_0(k_{\pm} r \sin\theta)(\nabla\phi)_{\perp}^2\ , \end{aligned} \tag{18}$$

and

$$\begin{aligned} A_{\parallel,\parallel} &= -\frac{1}{2} k_{\pm} \sin\theta \left[\frac{3}{2} N_{\pm}^2 \cos^2\theta - \frac{2\omega_e^4}{(\omega_e^2 - \Omega_e^2)(4\omega_e^2 - \Omega_e^2)} \times \left(N_{\pm}^2 \sin^2\theta - \frac{3}{4}\right) + \frac{1}{4} \frac{(7\omega_e^2 - \Omega_e^2)\Omega_e^2}{(\omega_e^2 - \Omega_e^2)(4\omega_e^2 - \Omega_e^2)} (\sin^2\theta \mp \rho)\right]\ , \\ A_{\parallel,\perp} &= [\omega_e^2/(\omega_e^2 - \Omega_e^2)] N_{\pm}^2 \sin\theta \cos\theta\ , \\ A_{\perp,\parallel} &= \frac{\omega_e^2}{\omega_e^2 - \Omega_e^2} k_{\pm} \cos\theta \left[N_{\pm}^2 \sin^2\theta + \frac{4\omega_e^2}{4\omega_e^2 - \Omega_e^2} \times \left(N_{\pm}^2 \sin^2\theta - \frac{3}{4}\right) - \frac{\Omega_e^2}{4\omega_e^2} \frac{6\omega_e^2 - \Omega_e^2}{4\omega_e^2 - \Omega_e^2} (\sin^2\theta \mp \rho)\right], \\ A_{\perp,\perp} &= \frac{1}{2} k_{\pm} \sin\theta \left[\left(\frac{1}{2} N_{\pm}^2 \cos^2\theta - \frac{6\omega_e^4}{(\omega_e^2 - \Omega_e^2)(4\omega_e^2 - \Omega_e^2)} \times \left(N_{\pm}^2 \sin^2\theta - \frac{3}{4}\right) + \frac{1}{4} \frac{(5\omega_e^2 - \Omega_e^2)\Omega_e^2}{(\omega_e^2 - \Omega_e^2)(4\omega_e^2 - \Omega_e^2)} (\sin^2\theta \mp \rho)\right)\right]. \end{aligned} \tag{19}$$

In addition,

$$\rho^2 = \sin^4\theta + 9(\omega_e^2/\Omega_e^2) \cos^2\theta$$

and

$$N_{\pm}^2 = 1 - \frac{3\omega_e^2}{12\omega_e^2 - 2\Omega_e^2(\sin^2\theta \mp \rho)}\ . \tag{20}$$

III. THE CASE OF QUASI-PLANAR SOLITONS

We choose to apply the expressions derived in Sec. II to the case of quasi-planar solitons since this is the only regime which is analytically accessible, and for which the angular spectrum of the radiation may be obtained in a relatively straightforward manner. For numerical simplicity, we assume the electric field and plasma density perturbations to be of the form[14]

$$\nabla\phi(r, z) = E(r)\, \mathrm{sech}[\kappa(r) z]\, \hat{\mathbf{e}}_z \tag{21}$$

and

$$\delta n(r, z)/n_0 = -6\kappa^2(r)\lambda_e^2\, \mathrm{sech}^2[\kappa(r) z]\ , \tag{22}$$

where λ_e is the electron Debye length, $1/\kappa(r)$ characterizes the parallel scale length of the perturbation,

$$W(r)/n_0 T_e = 12\kappa^2(r)\lambda_e^2(1 + \gamma_i T_i/T_e)\ , \tag{23}$$

$W(r) \equiv E^2(r)/8\pi$, T_i and γ_i are the ion temperature and ratio of specific heats, and T_e is the electron temperature. In addition, we assume that $\kappa(r) = \kappa_0 \exp(-r/\rho)$, where $\rho(\gg \kappa_0^{-1})$ characterizes the transverse scale length of the perturbation. It must be remarked that in order to neglect the radial component of the soliton field, we must have that

$$|\nabla_{\parallel}\phi| \gg [\omega_e^2/(\omega_e^2 - \Omega_e^2)]\, |\nabla_{\perp}\phi|\ , \tag{24}$$

and

$$J_0(k\kappa_0^{-1})\, |\nabla_{\parallel}\phi| \gg \tan(k\kappa_0^{-1}) J_1(k\kappa_0^{-1}) \frac{\omega_e^2}{\omega_e^2 - \Omega_e^2}\, |\nabla_{\perp}\phi|\ . \tag{25}$$

As a result, we cannot treat the case in which ω_e is arbitrarily close to Ω_e by means of Eqs. (21) and (22).

It should be pointed out that no analytic soliton solutions possessing cylindrical symmetry have presently been obtained for either field-free or uniformly magnetized plasmas. Indeed, it can be shown that Langmuir solitons in field-free plasmas are unstable against collapse in more than one dimension.[14-16] Therefore, in seeking to apply this work in the field-free limit, care must be exercised to restrict the application to early phases of the process in which the transverse extension of the soliton remains greater than the extension along the symmetry axis. The question of transverse collapse of multidimensional solitons in magnetized plasmas, however, still lacks solution. Studies of modulational instabilities (which constitute the Fourier space representation of the coordinate space collapse[17]) of Langmuir waves in a magnetized plasma indicate that the magnetic field can act to stabilize the solitons against transverse collapse.[18] In addition, results of numerical simulation[19] provide corroboration for the hypothesis that the magnetic field tends to inhibit transverse collapse. In short, while these structures may still undergo transverse collapse, the time scales may be greatly enhanced. In view of this, the assumption of quasi-steady-state soliton structures exhibiting greater transverse than parallel scale lengths is a reasonable one for magnetized plasmas, and the form of the radial dependence chosen in Eqs. (21)–(23) provides for analytical tractability without obscuring the physical characteristics of the emission.

Use of (21)–(23) immediately yields the following expressions for the emissivity at ω_e and $2\omega_e$:

$$\eta(\omega_e,\theta)=\frac{3\pi^2}{32}\left(\frac{v_e}{c}\right)^3\frac{n_0T_e}{\lambda_e}\left(\frac{\delta n_{ex}}{n_0}\right)^{-2}\left(1+\frac{\gamma_iT_i}{T_e}\right)^{-1}\sin^2\theta \times\sum_{\pm}\frac{N_\pm^5(\rho\pm\sin^2\theta)}{\rho}I_{1\pm}^2 \qquad (26)$$

and

$$\eta(2\omega_e,\theta)=\frac{\pi^2}{9}\left(\frac{v_e}{c}\right)^3\frac{n_0T_e}{\lambda_e}\tan^2\theta \times\sum_{\pm}\frac{N_\pm(\rho\pm\sin^2\theta)}{\rho}\Psi_\pm^2(\theta)I_{2\pm}^2\,, \qquad (27)$$

where $v_e^2\equiv T_e/m_e$,

$$\Psi_\pm(\theta)\equiv\frac{3}{2}N_\pm^2\cos^2\theta-\frac{2\omega_e^4}{(\omega_e^2-\Omega_e^2)(4\omega_e^2-\Omega_e^2)}\left(N_\pm^2\sin^2\theta-\frac{3}{4}\right) +\frac{7\omega_e^2\Omega_e^2}{4(\omega_e^2-\Omega_e^2)(4\omega_e^2-\Omega_e^2)}(\sin^2\theta\mp\rho)\,, \qquad (28)$$

and the source integrals are

$$I_{1\pm}\equiv\int_0^\infty dr\,r\left(1+\frac{k_\pm^2\cos^2\theta}{\kappa^2(r)}\right)\frac{W(r)}{n_0T_e} \times J_0(k_\pm r\sin\theta)\,\mathrm{sech}\left(\frac{\pi k_\pm\cos\theta}{2\kappa(r)}\right)\,, \qquad (29)$$

$$I_{2\pm}\equiv\int_0^\infty dr\,r\frac{k_\pm^2\cos\theta}{\kappa^2(r)}\frac{W(r)}{n_0T_e} \times J_0(k_\pm r\sin\theta)\,\mathrm{csch}\left(\frac{\pi k_\pm\cos\theta}{2\kappa(r)}\right)\,. \qquad (30)$$

The only regime in which Eqs. (29) and (30) can be integrated analytically is the case for which $k_\pm<\kappa_0$. However, it is important to recognize that $\eta(\omega_e,\theta)$ has been derived subject to the condition that $k_\pm>\kappa_0$ (which is equivalent to $6v_e^2N_\pm^2>c^2\,|\delta n_{ex}/n_0|$) and no further analytic reduction is possible for the case of emission at ω_e[Eq. (29)]. Turning, therefore, to the case of emission at $2\omega_e$, we find that when

$$6(v_e/c)^2\ll|\delta n_{ex}/n_0|\,, \qquad (31)$$

the emissivity becomes

$$\eta(2\omega_e,\theta)=\frac{16}{3}\left(\frac{v_e}{c}\right)^4\frac{c^3W_0}{\omega_e^3}\left(1+\frac{\gamma_iT_i}{T_e}\right)\sum_{\pm}\Theta_\pm(\theta)\,, \qquad (32)$$

where $W_0\equiv W(r=0)$ and

$$\Theta_\pm(\theta)=\frac{\rho\pm\sin^2\theta}{\rho N_\pm}\Psi_\pm^2(\theta)\frac{k_\pm^4\rho^4\tan^2\theta}{(1+k_\pm^2\rho^2\sin^2\theta)^3}\,. \qquad (33)$$

Thus, $\eta(2\omega_e,\theta)\sim k_\pm^4\rho^4$ in the limit in which $k_\pm\rho<1$, and $\eta(2\omega_e,\theta)\sim(k_\pm\rho)^{-2}$ in the opposite case.

The emissivity expressed in Eqs. (32) and (33) admits of relatively simple numerical analysis, and we display the angular spectra of the ordinary and extraordinary modes in Figs. 1 and 2 by plotting $\Theta_\pm(\theta)$ for $0\le\theta\le\pi/2$ and several choices of ω_e/Ω_e and $\omega_e\rho/c$. As shown in the figures, there is a quadrupole radiation pattern for both the ordinary and extraordinary modes, which is highly sensitive to the transverse scale size of the soliton. Specifically, for $\omega_e\rho/c>1$, the radiation is strongly beamed in the directions parallel and antiparallel to soliton propagation, and for $\omega_e\rho/c\lesssim1$, peak emission

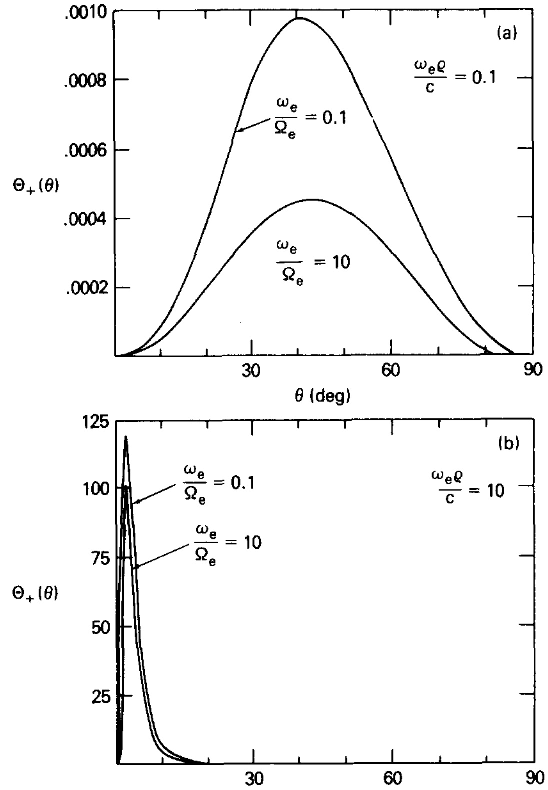

FIG. 1. Plots of $\Theta_+(\theta)$ for $\omega_e/\Omega_e=0.1$, 10 and (a) $\omega_e\rho/c=0.1$ and (b) $\omega_e\rho/c=10$.

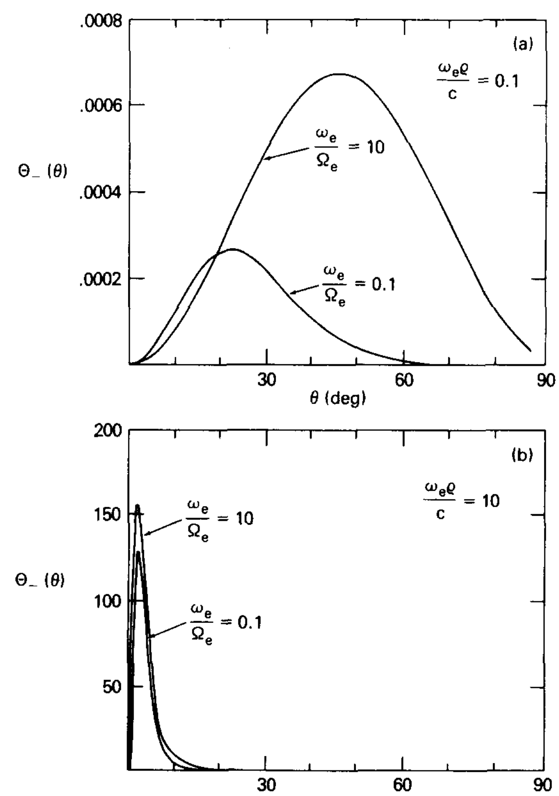

FIG. 2. Plots of $\Theta_-(\theta)$ for $\omega_e/\Omega_e=0.1$, 10 and (a) $\omega_e\rho/c=0.1$ and (b) $\omega_e\rho/c=10$.

occurs for $\theta \lesssim \pi/4$. We note that for $\omega_e \rho/c < 1$, the emissivity scales as $\eta(2\omega_e, \theta) \sim \sin^2 2\theta$ in the limit of large[2,4] ω_e/Ω_e. In Fig. 3, we plot $\eta_+(2\omega_e, \theta)/\eta_-(2\omega_e, \theta)$ for parameters consistent with those used in the computations of Fig. 1. We remark that this quantity appears to be relatively insensitive to ρ over the range studied (i.e., $0.1 < \omega_e\rho/c < 10$), and we display the result for $\omega_e\rho/c = 1$. The principal results are (i) for $\omega_e/\Omega_e < 1$, the ordinary mode tends to dominate the emission, but that this situation is reversed when $\omega_e/\Omega_e > 1$ and (ii) the characteristic dominance of either mode is greatly enhanced for $\theta \gtrsim 40°$.

Finally, we observe that condition (31) is equivalent to the requirement that $W_0/n_0T_e \gg 12(v_e/c)^2$, and it is clear from Eq. (32) that in this limit, $\eta(2\omega_e, \theta) \sim W_0$.

In order to treat the case in which $k_\pm < \kappa_0$, we must rely on wholly numerical methods. To this end we first rewrite

$$\eta(\omega_e, \theta) = \frac{3\pi^2}{8}\left(\frac{v_e}{c}\right)^2 \frac{c^3}{\omega_e^3} n_0 T_e \left(1 + \frac{\gamma_i T_i}{T_e}\right) \sum_\pm P_\pm^{(1)} \tag{34}$$

and

$$\eta(2\omega_e, \theta) = \left(\frac{48\pi}{3}\right)^2 \left(\frac{v_e}{c}\right)^6 \frac{c^3}{\omega_e^3} n_0 T_e \left(1 + \frac{\gamma_i T_i}{T_e}\right)^2 \sum_\pm P_\pm^{(2)}, \tag{35}$$

where

$$P_\pm^{(1)} \equiv \frac{N_\pm^5 \sin^2\theta(\rho \pm \sin^2\theta)}{\rho} \left\{ \int_0^\infty dx\, x J_0(N_\pm x \sin\theta) \times \left[\exp\left(-\frac{2xc}{\omega_e\rho}\right) + \frac{k_\pm^2 \cos^2\theta}{\kappa_0^2}\right] \operatorname{sech}\left[\frac{\pi k_\pm \cos\theta}{2\kappa_0} \exp\left(\frac{xc}{\omega_e\rho}\right)\right]\right\}^2 \tag{36}$$

and

$$P_\pm^{(2)} \equiv \frac{N_\pm^5 \sin^2\theta(\rho \pm \sin^2\theta)}{\rho} \Psi_\pm^2(\theta) \left\{ \int_0^\infty dx\, x J_0(N_\pm x \sin\theta) \times \operatorname{csch}\left[\frac{\pi k_\pm \cos\theta}{2\kappa_0} \exp\left(\frac{xc}{\omega_e\rho}\right)\right]\right\}^2. \tag{37}$$

The dependence of the emissivity at $\omega \simeq \omega_e$ on both θ and W_0 are contained in $P_\pm^{(1,2)}$, and it is these quantities that we evaluate here. Note, again, that the condition required for the validity of Eq. (34) is that

$$6N_\pm(v_e/c) > \kappa_0\lambda_e \sim (W_0/n_0T_e)^{1/2},$$

and that $N_+^2 \sim |\delta n_{ex}/n_0| \sim W_0/n_0T_e$ while $N_-^2 \sim 1$. Thus, it is difficult to satisfy this requirement for ordinary mode waves, and we restrict the analysis to consideration of the extraordinary mode emissivity at $\omega \simeq \omega_e$ (no restriction is necessary for emission at $2\omega_e$). It is the existence of an electromagnetic mode with frequency $\omega \simeq \omega_e$ and index of refraction $N \sim 1$ which constitutes a major distinction between magnetized and unmagnetized plasmas.

We consider the case of emission at $\omega \simeq \omega_e$ first, and plot the results of the numerical integration of $P_-^{(1)}$ vs W_0/n_0T_e in Fig. 4. It should be noted, again, that the constraints on the analysis in this frequency regime are that $N_-(v_e/c) > \kappa_0\lambda_e > \lambda_e/\rho$. In the results presented, we chose $T_e = 0.1$ keV and $\omega_e\rho/c = 10$, which imply that $\rho \simeq 715\lambda_e$ and $1.4\times10^{-3} \lesssim \kappa_0\lambda_e \lesssim 1.4\times10^{-2}$. For simplicity,

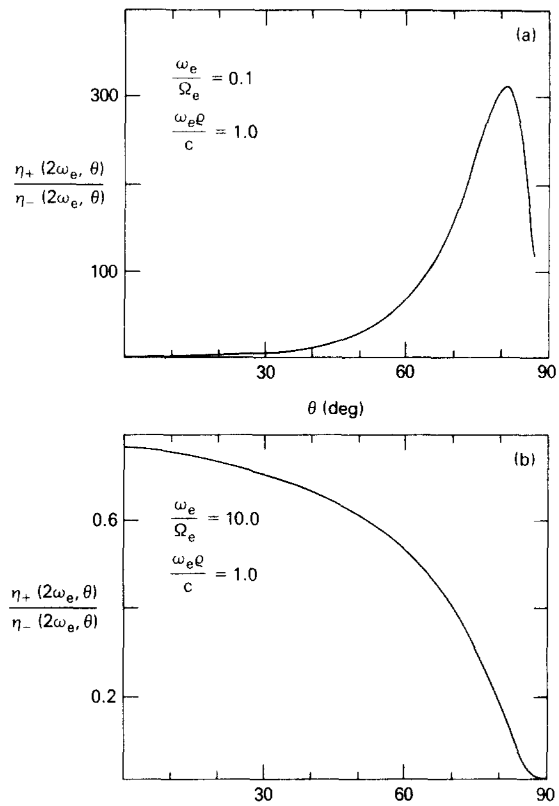

FIG. 3. Plots of $\eta_+(2\omega_e, \theta)/\eta_-(2\omega_e, \theta)$ for $\omega_e\rho/c = 1$ and (a) $\omega_e/\Omega_e = 0.1$ and (b) $\omega_e/\Omega_e = 10$.

we have assumed that $T_i = 0$ in the analysis. It is clear from the figure that the angular spectrum depends critically on both ω_e/Ω_e and the soliton amplitude, and that no simple scaling law can be found between $\eta_-(\omega_e, \theta)$ and W_0/n_0T_e. It should be observed, however, that while increases in plasma density (i.e., in ω_e/Ω_e) leave the scaling at low levels of soliton amplitude relatively unchanged, the scaling of the emissivity with W_0/n_0T_e and

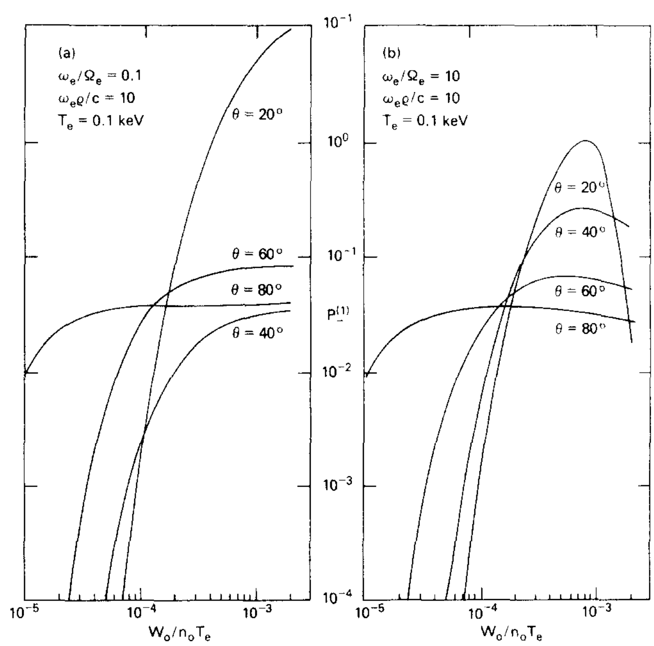

FIG. 4. Plots of $P_-^{(1)}$ vs W_0/n_0T_e for $T_e = 0.1$ keV, $\omega_e\rho/c = 10$, and (a) $\omega_e/\Omega_e = 0.1$ and (b) $\omega_e/\Omega_e = 10$.

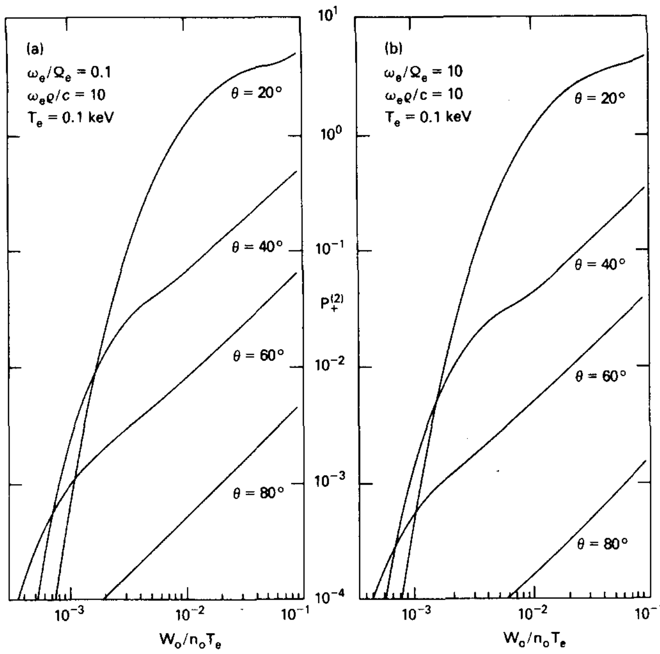

FIG. 5. Plots of $P_+^{(2)}$ vs W_0/n_0T_e for $T_e = 0.1$ keV, $\omega_e\rho/c = 10$, and (a) $\omega_e/\Omega_e = 0.1$ and (b) $\omega_e/\Omega_e = 10$.

the angular spectrum of the emission are substantially altered at higher levels of W_0.

In Figs. 5 and 6 we plot the results of a numerical integration of $P_+^{(2)}$ and $P_-^{(2)}$ vs W_0/n_0T_e, respectively. It is clear from both figures that (i) as W_0 increases we recover the result in Eq. (32) in which $\eta_\pm(2\omega_e, \theta) \sim W_0$, (ii) at lower levels of W_0 the emissivity increases faster than W_0, and (iii) the angular spectrum of the emission is sensitive to the soliton amplitude. We also observe that while the ordinary mode emissivity (i.e., $P_+^{(2)}$) is relatively insensitive to the plasma density, the ex-

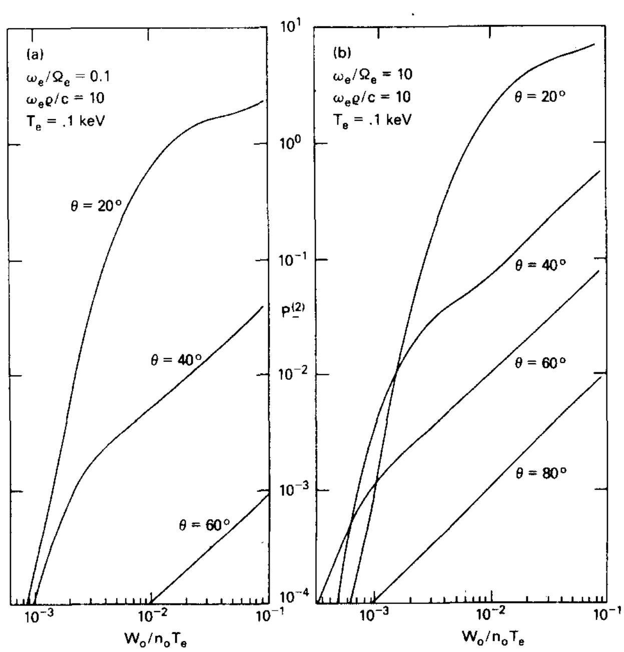

FIG. 6. Plots of $P_-^{(2)}$ vs W_0/n_0T_e for $T_e = 0.1$ keV, $\omega_e\rho/c = 10$, and (a) $\omega_e/\Omega_e = 0.1$ and (b) $\omega_e/\Omega_e = 10$.

traordinary mode emissivity is greatly modified in going from $\omega_e/\Omega_e = 0.1$ to $\omega_e/\Omega_e = 10$. This can be explained by noting that the emission is in the slow extraordinary mode for $\omega_e/\Omega_e = 0.1$ (i.e., the emission frequency is below the upper hybrid frequency), and in the fast extraordinary mode for $\omega_e/\Omega_e = 10$. This will have severe consequences for the radiation observed from outside the plasma, since slow extraordinary mode waves cannot readily escape from the plasma without tunneling through the upper hybrid layer or mode coupling to the ordinary or fast extraordinary modes. Finally, we remark that, as shown in (32), the angular spectrum of the radiation and the scaling of $\eta_\pm(2\omega_e, \theta)$ with W_0 should also be sensitive to the transverse dimension of the soliton; however, it is beyond the scope of this work to treat this scaling in the regime in which $k_\pm > \kappa_0$.

IV. SUMMARY AND DISCUSSION

In this work, expressions have been derived for the radiation of an arbitrary three-dimensional Langmuir wave packet at $\omega \simeq \omega_e$ and $2\omega_e$ in a uniformly magnetized plasma. The analysis of the radiation at the plasma frequency has been limited to the regime in which the radiation wavelength is much less than the scale length of the soliton in the interest of deriving an analytic expression for the emissivity, which imposes the requirement that $12(v_e/c)^2 N_\pm^2 \gtrsim W_0/n_0T_e$. Since the only electromagnetic mode in a field-free plasma with frequency $\omega \simeq \omega_e$ has an index of refraction $N^2 \sim W_0/n_0T_e$, this condition imposes a severe restriction on the present analysis to that of a very hot plasma. However, the presence of an ambient magnetic field introduces an additional mode with a mixed electrostatic/electromagnetic polarization (i.e., the extraordinary mode) having an index of refraction of the order of unity in the vicinity of the plasma frequency and which presents no such severe restriction. While these waves cannot readily escape from the plasma (unless some means of tunneling through the upper hybrid layer or mode conversion to the ordinary or fast extraordinary modes is possible) and should not be an important characteristic of radiation from astrophysical plasmas, study of this radiation mode may be important in laboratory plasmas.[6-10]

In order to determine relatively simple expressions for the radiation emissivity and, thereby, to determine the angular spectrum of the emission as well as the scaling of the radiated power with soliton amplitude, the specific case of one-dimensional Langmuir solitons has been studied in some depth. In this limiting regime, it is shown that the angular spectrum is sensitive to both the soliton amplitude and to the transverse scale size of the soliton. While no simple scaling law between the emissivity and the soliton amplitude is readily apparent for $\omega \simeq \omega_e$, it is clear that for $\omega \simeq 2\omega_e$ the emissivity is linearly proportional to the soliton amplitude when W_0 exceeds a certain threshold which depends on the plasma density, the ambient magnetic field, and the angle of propagation of the radiation. The immediate significance of this result is to the scaling of the second harmonic radiation in type III solar bursts.[2]

ACKNOWLEDGMENTS

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APPENDIX: THE DISPERSION TENSOR AND SOURCE CURRENT

The fluctuation fields giving rise to the emission are of high order in $\nabla\phi$, and in order to treat emission at ω_e and $2\omega_e$ we must solve the equations

$$\frac{\partial}{\partial t}\delta\mathbf{v} = -\frac{e}{m_e}\delta\mathbf{E} - \mathbf{v}^{(1)}\cdot\nabla\mathbf{v}^{(1)} + \Omega_e\hat{\mathbf{e}}_z\times\delta\mathbf{v}\,, \tag{A1}$$

$$\nabla\times\delta\mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}\delta\mathbf{E} - \frac{4\pi e}{c^2}[(n_0+\delta n)\delta\mathbf{v} + (n^{(1)}+\delta n)\mathbf{v}^{(1)}]\,, \tag{A2}$$

$$\nabla\times\delta\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\delta\mathbf{B}\,, \tag{A3}$$

where $\delta\mathbf{E}$ and $\delta\mathbf{B}$ are the radiation fields, $\delta\mathbf{v}$ is the high-order velocity fluctuation, and $n^{(1)}$ and $\mathbf{v}^{(1)}$ are the first-order density and velocity fluctuations. Note that δn, which describes the caviton structure, is itself of second order in $\nabla\phi$. Thus, the term in $\delta n\ \delta\mathbf{v}$ is of at least fourth order in $\nabla\phi$ and gives rise to the turbulent shift in the plasma frequency. In addition, the term in $\delta n\mathbf{v}^{(1)}$ is of third order in $\nabla\phi$ and is responsible for the oscillatory current at ω_e. In contrast, the terms giving rise to emission at $2\omega_e$ are of second order in $\nabla\phi$.

Eliminating $\delta\mathbf{B}$ from this system of equations, we find, after some straightforward manipulations, that

$$\left(\frac{c}{\omega^2}(\mathbf{kk}-k^2\mathsf{I}) + \mathsf{I} - \sigma\right)\cdot\delta\mathbf{E}(\mathbf{k},\omega) = \frac{m_e}{e}\sigma\cdot\left(\mathbf{v}^{(1)}\cdot\nabla\mathbf{v}^{(1)}\right)_{\mathbf{k},\omega} + \frac{4\pi ie}{\omega}(\delta n\delta\mathbf{v})_{\mathbf{k},\omega} - \frac{4\pi e}{\omega^2}\left(\frac{\partial}{\partial t}(n^{(1)}+\delta n)\mathbf{v}^{(1)}\right)_{\mathbf{k},\omega} \tag{A4}$$

and

$$\delta\mathbf{v}(\mathbf{k},\omega) = -\frac{i\omega}{4\pi en_0}\sigma\cdot\left(\delta\mathbf{E}(\mathbf{k},\omega) + \frac{m_e}{e}(\mathbf{v}^{(1)}\cdot\nabla\mathbf{v}^{(1)})_{\mathbf{k},\omega}\right), \tag{A5}$$

where $(\ )_{\mathbf{k},\omega}$ denotes the Fourier transform of the enclosed quantity,

$$\sigma_{xx} = \sigma_{yy} = \frac{\omega_e^2}{(\omega^2-\Omega_e^2)}\,,\quad \sigma_{zz} = 1 - \frac{\omega_e^2}{\omega^2}\,,$$

$$\sigma_{xy} = -\sigma_{yx} = -i\frac{\omega_e^2\Omega_e}{\omega(\omega^2-\Omega_e^2)}\,,\ \text{and}\ \sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0\,.$$

In order to evaluate the convolution in $\delta n\delta\mathbf{v}$, we assume a narrow spectrum of the form

$$\delta\mathbf{v}(\mathbf{k},\omega) = \delta\mathbf{v}(\mathbf{k}_0,\omega_0)\delta(\mathbf{k}-\mathbf{k}_0)\delta(\omega-\omega_0)\,.$$

It follows, therefore, that if $k \gg |\nabla\phi/\phi|$, then

$$(\delta n\delta\mathbf{v})_{\mathbf{k},\omega} \simeq \delta n\delta\mathbf{v}(\mathbf{k},\omega)\,. \tag{A6}$$

Combination of (A4)–(A6) then yields Eq. (5), in which the source current is

$$\delta\mathbf{J}_s(\mathbf{k},\omega) = \frac{i\omega}{4\pi}\left[\frac{m_e}{e}\sigma\cdot(\mathbf{v}^{(1)}\cdot\nabla\mathbf{v}^{(1)})_{\mathbf{k},\omega} - \frac{4\pi e}{\omega^2}\times\left(\frac{\partial}{\partial t}(n^{(1)}+\delta n)\mathbf{v}^{(1)}\right)_{\mathbf{k},\omega}\right]. \tag{A7}$$

The first-order density and velocity fluctuations satisfy the equations

$$\frac{\partial}{\partial t}n^{(1)} + n_0\nabla\cdot\mathbf{v}^{(1)} = 0 \tag{A8}$$

and

$$\frac{\partial}{\partial t}\mathbf{v}^{(1)} = -\frac{e}{m_e}\nabla\phi\sin\omega_e t + \Omega_e\hat{\mathbf{e}}_z\times\mathbf{v}^{(1)}\,. \tag{A9}$$

The solutions to (A8) and (A9) follow immediately:

$$n^{(1)} = -\frac{en_0}{m_e\omega_e^2}\left(\nabla^2\phi + \frac{\Omega_e^2}{\omega_e^2-\Omega_e^2}\nabla_\perp^2\phi\right)\sin\omega_e t\,, \tag{A10}$$

$$\mathbf{v}^{(1)} = \frac{e}{m_e\omega_e}\left[\left(\nabla\phi + \frac{\Omega_e^2}{\omega_e^2-\Omega_e^2}\nabla_\perp\phi\right)\cos\omega_e t + \frac{\omega_e\Omega_e}{\omega_e^2-\Omega_e^2}(\hat{\mathbf{e}}_z\times\nabla\phi)\sin\omega_e t\right]. \tag{A11}$$

Substitution of (A10) and (A11) into (A7) reproduces Eqs. (10) and (11).

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