relativistic kinetic energy. Expanding the square root, we get

$$(p^2/2\mu - p^4/8\mu^3c^2 + \cdots + V)\psi = \epsilon\psi, \qquad (9)$$

where we have spelled out only the lowest relativistic term. With $V = \frac{1}{2}kr^2$, we have here the nonrelativistic harmonic-oscillator problem with a p^4 term as the leading perturbation. From the usual x and p_x matrix elements we deduce that $\langle p^4 \rangle_0 = \mu^4\omega^4\langle r^4 \rangle_0$, whereupon use of Eq. (6) yields Eq. (7).

The latter method is clearly simpler when only the first correction is required. However, going beyond that, there is in Eq. (9) an increasing number of perturbation terms, to be treated to increasing orders of perturbation theory. In the Klein–Gordon solution, on the other hand, there is only the r^4 perturbation term, which is easily handled by higher-order perturbation theory or by matrix diagonalization; increasing the accuracy in expanding Eq. (4) is trivial. The wave functions resulting from such a treatment will be of the form $\sum_N c_N | Nlm \rangle_0$. We conclude that the Klein–Gordon solution is generally preferable to the expansion in p^2 in that the former provides a clear scheme for higher approximations. However, the limited applicability of a one-particle theory must be remembered in making higher approximations.

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* On leave from the University of Helsinki, Finland.

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6 The harmonic-oscillator wave functions extend to infinity and include the region where $V > E$. There the probability density for the Klein–Gordon equation is negative, indicating the general insufficiency of a one-particle theory; see Ref. 4 and S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row Publishers, Inc., New York, 1961), pp. 63–64. (Schweber's discussion of the Coulomb field is wrong; for $E > 0$ and $V < 0$, the probability density is positive everywhere.) However, we have already effectively excluded the far region.

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Bremsstrahlung Radiation in Plasmas

Konstantinos Papadopoulos

Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742

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A new, short derivation of the radiation formulas from a homogeneous, field free, nonrelativistic plasma is presented. The nonrelativistic assumptions are emphasized.

In this note we present a derivation of the formulas for the bremsstrahlung radiation emitted from a nonrelativistic plasma. These formulas have been previously presented by Dupree[1] and by Birmingham *et al.*[2] However, their derivations have the disadvantage of being very complicated and lengthy. An alternative method is presented here, using a technique which was previously[3] applied in deriving the relativistically correct emission power. This derivation besides being short has the advantage of clearly showing the necessary nonrelativistic assumptions.

Dawson *et al.*[4] recently proposed a method of carrying out plasma kinetic theory calculations by expanding the singular distribution function f^α in terms of the deflections of the particles from their non-interacting orbits. Thus if

$$f^\alpha = \langle f^\alpha \rangle + \delta f^\alpha,$$

where $\langle f^\alpha \rangle$ is the slowly time varying ensemble average and δf^α the fluctuations about $\langle f^\alpha \rangle$, we directly expand (same notation as Refs. 3 and 4).

$$\delta f^\alpha = f_1{}^\alpha + f_2{}^\alpha + \cdots$$
$$\delta \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots$$
$$\delta \mathbf{B}_1 = \mathbf{B}_1 + \mathbf{B}_2 + \cdots.$$

Assuming $\langle \mathbf{E} \rangle = \langle \mathbf{B} \rangle = 0$, and $f_0{}^\alpha$ isotropic, the second order equations are[5]

$$[(\partial/\partial t) + \mathbf{v}\cdot(\partial/\partial \mathbf{r})] R_2{}^\alpha(\mathbf{r}, \mathbf{p}, t) + q_\alpha \mathbf{E}_2 \cdot (\partial f_0{}^\alpha/\partial \mathbf{p}) = 0$$

$$\nabla \times \mathbf{E}_2 + c^{-1}(\partial \mathbf{B}_2/\partial t) = 0 \tag{1}$$

$$\nabla \times \mathbf{B}_2 - c^{-1}(\partial \mathbf{E}_2/\partial t) - (4\pi/c) \sum_\alpha q_\alpha \int d\mathbf{p} R_2{}^\alpha(\mathbf{r}, \mathbf{p}, t)\mathbf{v} = (4\pi/c)\mathbf{j}_s(\mathbf{r}, t)$$

$$[(\partial/\partial t) + \mathbf{v}\cdot(\partial/\partial \mathbf{r})] \Psi_2{}^\alpha(\mathbf{r}, \mathbf{p}, t) = -q_\alpha\{[\mathbf{E}_1 + (\mathbf{v} \times \mathbf{B}_1/c)]\cdot(\partial f_1{}^\alpha/\partial \mathbf{p}) - \langle\ \rangle\} \tag{2}$$

$$\mathbf{j}_s(\mathbf{r}, t) = \sum_\alpha q_\alpha \int d\mathbf{p} \Psi_2{}^\alpha(r, \mathbf{p}, t)\mathbf{v}, \tag{3}$$

where

$$f_2(r, \mathbf{p}, t) = \Psi_2{}^\alpha(\mathbf{r}, \mathbf{p}, t) + R_2{}^\alpha(\mathbf{r}, \mathbf{p}, t).$$

It can be seen from above that $\psi_2{}^\alpha$ is produced by the direct interaction of the first order fluctuations with the first order fields, and $R_2{}^\alpha$ simply represents the shielding of these fluctuations.

The rate at which power is radiated into transverse modes is equal to the rate at which work is done by transverse current fluctuations on the transverse field. Solving Eq. (1) for $\mathbf{E}_2$ one thus finds for the emission spectrum,[3]

$$dI/d\Omega = [|\Omega|(\Omega^2 - \omega_e{}^2)^{1/2}/8\pi^5 c^3] P(K, \Omega), \tag{4}$$

where

$$\Omega^2 \simeq \omega_e{}^2 + K^2 c^2,$$

and

$$P(k, \omega) = \sum_{\alpha,\beta} \int d\mathbf{k}'' d\omega'' \times \exp i[(\mathbf{k} - \mathbf{k}'')\cdot\mathbf{X} - (\omega - \omega'')T] \times \langle j_{s\perp}{}^\alpha(\mathbf{k}, \omega) j_{s\perp}{}^\beta(\mathbf{k}'', \omega'')\rangle. \tag{5}$$

The symbol $\perp$ means perpendicular to $\mathbf{k}$.

We calculate next $P(k, \omega)$. From Eqs. (2) and (3) and keeping only the longitudinal fields,[6]

$$\mathbf{j}_s{}^\alpha(\mathbf{k}, \omega) = -iq_\alpha{}^2 \int d\mathbf{p}[\mathbf{v}/(\omega - \mathbf{k}\cdot\mathbf{v})] \int d\mathbf{k}' \int d\omega' \times \mathbf{E}_1(\mathbf{k}', \omega')\cdot(\partial/\partial \mathbf{p}) f_1{}^\alpha(\mathbf{k} - \mathbf{k}', \omega - \omega', \mathbf{p}).$$

Integrating by parts and noticing that

$$\mathbf{E}_1(\mathbf{k}', \omega') = -(4\pi i \mathbf{k}'/k'^2) \sum_\gamma q_\gamma n_1{}^\gamma(\mathbf{k}', \omega'),$$

we obtain

$$(2\pi)^4 \mathbf{j}_s{}^\alpha(\mathbf{k}, \omega) = \sum_\gamma 4\pi q_\alpha{}^2 q_\gamma \int (d\mathbf{k}'/k'^2) \int d\omega' n_1{}^\gamma(\mathbf{k}', \omega') \times \int d\mathbf{p} f_1{}^\alpha(\mathbf{k} - \mathbf{k}', \omega - \omega', \mathbf{p})\mathbf{k}' \cdot(\partial/\partial \mathbf{p})[\mathbf{v}/(\omega - \mathbf{k}\cdot\mathbf{v})]. \tag{6}$$

For a nonrelativistic plasma we have that

(i) Since ω, k are the emitted transverse frequency and wave-vector $(\omega/k) \sim O(c)$. Then one can assume

$$(\omega - \mathbf{k}\cdot\mathbf{v})^{-1} = \omega^{-1}\{1 + (\mathbf{k}\cdot\mathbf{v}/\omega) + [(\mathbf{k}\cdot\mathbf{v})^2/\omega^2] + \cdots\} \approx \omega^{-1}. \tag{7}$$

The term $\mathbf{k}\cdot\mathbf{v}/\omega$ vanishes on performing angular integrations for isotropic f_0's, so that the above approximation corresponds to neglect of $(v/c)^2$ terms.

(ii) Assume

$$\mathbf{k}'\cdot(\partial/\partial \mathbf{p}) \approx 1/m_\alpha \mathbf{k}'\cdot(\partial/\partial \mathbf{v}). \tag{8}$$

This is also equivalent to neglecting $(v/c)^2$ terms.

It is obvious that if relativistic particles are present the above approximations cannot be used.

From Eqs. (6), (7), and (8),

$$(2\pi)^4 \mathbf{j}_s{}^\alpha(\mathbf{k}, \omega) = \sum_\gamma (\omega_\alpha{}^2/\omega) q_\gamma \int d\mathbf{k}'(\mathbf{k}'/k'^2) \int d\omega' n_1{}^\gamma(\mathbf{k}', \omega') n_1{}^\alpha(\mathbf{k} - \mathbf{k}', \omega - \omega').$$

Using this value of $\mathbf{j}_s{}^\alpha(\mathbf{k}, \omega)$ in Eq. (5),

$$(2\pi)^8 P(\mathbf{k}, \omega) = \sum_{\alpha,\beta,\gamma,\delta} \omega_\alpha{}^2 \omega_\beta{}^2 q_\delta q_\gamma \int d\mathbf{k}' d\omega' d\mathbf{k}'' d\omega'' d\mathbf{k}''' d\omega''' \times \exp[i(\mathbf{k} - \mathbf{k}'')\cdot\mathbf{X} - i(\omega - \omega'')T](k_\perp{}'/k'^2)(k_\perp{}'''/k'''^2)(\omega\omega'')^{-1} \times \langle n_1{}^\alpha(\mathbf{k} - \mathbf{k}', \omega - \omega') n_1{}^\gamma(\mathbf{k}', \omega') n_1{}^\beta(\mathbf{k}'', \omega'') n_1{}^\delta(\mathbf{k}'' - \mathbf{k}''', \omega'' - \omega''')\rangle. \tag{9}$$

For a homogeneous plasma

$$\langle n_1^\alpha(\mathbf{k}_1,\omega_1)n_1^\beta(\mathbf{k}_2,\omega_2)n_1^\gamma(\mathbf{k}_3,\omega_3)n_1^\delta(\mathbf{k}_4,\omega_4)\rangle=(2\pi)^8[\langle n_1^\alpha n_1^\beta\mid\mathbf{k}_1,\omega_1\rangle\langle n_1^\gamma n_1^\delta\mid\mathbf{k}_3,\omega_3\rangle\delta(\mathbf{k}_1-\mathbf{k}_2)\delta(\omega_1-\omega_2)$$
$$\times\delta(\mathbf{k}_3-\mathbf{k}_4)\delta(\omega_3-\omega_4)+\langle n_1^\alpha n_1^\gamma\mid\mathbf{k}_1,\omega_1\rangle\langle n_1^\beta n_1^\delta\mid\mathbf{k}_2,\omega_2\rangle\delta(\mathbf{k}_1-\mathbf{k}_3)\delta(\omega_1-\omega_3)\delta(\mathbf{k}_2-\mathbf{k}_4)\delta(\omega_2-\omega_4)$$
$$+\langle n_1^\alpha n_1^\delta\mid\mathbf{k}_1,\omega_1\rangle\langle n_1^\beta n_1^\delta\mid\mathbf{k}_2,\omega_2\rangle\delta(\mathbf{k}_1-\mathbf{k}_4)\delta(\omega_1-\omega_4)\delta(\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_2-\omega_3)]+O(\epsilon^3). \quad (10)$$

Using Eq. (10) in Eq. (9), carrying out the integrations and substituting for $P(K,\Omega)$ in Eq. (4), we find

$$\frac{dI}{d\Omega}=\frac{|\Omega|(\Omega^2-\omega_e^2)^{1/2}}{(2\pi)^4c^3}\frac{1}{\Omega^2}\sum_{\alpha,\beta,\gamma,\delta}\omega_\alpha^2\omega_\beta^2q_\gamma q_\delta$$
$$\times\int\frac{d\mathbf{k}}{k^2}\int d\omega\left[\frac{|\mathbf{k}\cdot\boldsymbol{\varepsilon}|^2}{k^2}S^{\alpha\beta}(\mathbf{k},\omega)S^{\gamma\delta}(\mathbf{K}-\mathbf{k},\Omega-\omega)\right.$$
$$\left.+(\mathbf{k}\cdot\boldsymbol{\varepsilon}^*)\frac{(\mathbf{K}-\mathbf{k})\cdot\boldsymbol{\varepsilon}}{|\mathbf{K}-\mathbf{k}|^2}S^{\beta\gamma}(\mathbf{k},\omega)S^{\alpha\delta}(\mathbf{K}-\mathbf{k},\Omega-\omega)\right],$$

where $S^{\alpha\beta}(\mathbf{k},\omega)=\langle n_1^\alpha n_1^\beta\mid\mathbf{k},\omega\rangle$ and $\boldsymbol{\varepsilon}$ is the polarization vector with normalization $\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^*=2\pi$. This is the emission formula as given by Tidman and Dupree.[7]

The functions $S^{\alpha\beta}$ are the first-order density correlations. They are easily calculated using Rostoker's dressed particle model. The first-order particle density perturbation is given by[8]

$$n_1^\alpha(\mathbf{k},\omega)=n_0^\alpha(\mathbf{k},\omega)$$
$$+[L^\alpha(\mathbf{k},\omega)/\epsilon_l(\mathbf{k},\omega)]\sum_\gamma(q_\gamma/q_\alpha)n_0^\gamma(\mathbf{k},\omega), \quad (11)$$

where $n_0^\alpha(\mathbf{k},\omega)$ is the Fourier transform of the singular particle density $\delta(\mathbf{x}-\mathbf{x}_{0\alpha}-\mathbf{v}_\alpha t)$ moving in its unperturbed orbit and

$$L^\alpha(\mathbf{k},\omega)=\frac{\omega_e^2}{k^2}\int d\mathbf{v}\frac{\mathbf{k}\cdot(\partial f_0^\alpha/\partial\mathbf{v})}{\mathbf{k}\cdot\mathbf{v}-\omega}$$

$$\epsilon_l(\mathbf{k},\omega)=1-\sum_\alpha L^\alpha(\mathbf{k},\omega).$$

From the above equations, since $S^{\alpha\beta}(\mathbf{k},\omega)=\langle n_1^\alpha n_1^{*\beta}\mid\mathbf{k},\omega\rangle$,

$$(2\pi)^{-1}S^{\alpha\beta}(\mathbf{k},\omega)=\delta(\alpha,\beta)\int d\mathbf{v}f_0^\alpha(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$
$$+\frac{L^\alpha}{\epsilon_l}\int d\mathbf{v}f_0^\beta(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$
$$+\frac{L^{*\beta}}{\epsilon_l^*}\int d\mathbf{v}f_0^\alpha(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$
$$+\frac{L^\alpha L^\beta}{|\epsilon_l|^2}\sum_\delta\frac{q_\delta^2}{q_\alpha q_\beta}\int d\mathbf{v}f_0^\delta(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v}). \quad (12)$$

In deriving Eq. (12), use was made of the fact that for a homogeneous plasma

$$\langle n_0^\alpha(\mathbf{r}_1,t_1)n_0^\beta(\mathbf{r}_2,t_2)\rangle=\delta(\alpha,\beta)\int d\mathbf{v}f_0^\alpha(\mathbf{v})\times\delta[\mathbf{r}_1-\mathbf{r}_2-\mathbf{v}(t_1-t_2)],$$

which results in

$$\langle n_0^\alpha n_0^\beta\mid\mathbf{k},\omega\rangle=2n\delta(\alpha,\beta)\int d\mathbf{v}f_0^\alpha(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v}).$$

Equations (11) and (12) are the final formulae for the plasma bremsstrahlung radiation in terms of the particle distribution functions.

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