relativistic kinetic energy. Expanding the square root, we get

$$\left(\frac{p^2}{2\mu} - p^4/8\mu v^2 + \cdots + V\right)\psi = c\psi,$$

(9)

where we have spelled out only the lowest relativistic term. With \( V = \frac{1}{2} k r^2 \), we have here the nonrelativistic harmonic-oscillator problem with a \( p^4 \) term as the leading perturbation. From the usual \( x \) and \( p \) matrix elements we deduce that

$$\langle p^4 \rangle_n = \mu^2 \frac{d^4}{d^4}(r^4)_n,$$

whereupon use of Eq. (6) yields Eq. (7).

The latter method is clearly simpler when only the first correction is required. However, going beyond that, there is in Eq. (9) an increasing number of perturbation terms, to be treated to increasing orders of perturbation theory. In the Klein–Gordon solution, on the other hand, there is only the \( r^4 \) perturbation term, which is easily handled by higher-order perturbation theory or by matrix diagonalization; increasing the accuracy in expanding Eq. (4) is trivial. The wave functions resulting from such a treatment will be of the form \( \sum_N c_N | N n \rangle \). We conclude that the Klein–Gordon solution is generally preferable to the expansion in \( p^2 \) in that the former provides a clear scheme for higher approximations. However, the limited applicability of a one-particle theory must be remembered in making higher approximations.

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* On leave from the University of Helsinki, Finland.

1 C. Fronsdal and A. Mondragón, private communication.


5 It is tacitly assumed that \( k \) is a Lorentz scalar.

6 The harmonic-oscillator wave functions extend to infinity and include the region where \( V > E \). There the probability density for the Klein-Gordon equation is negative, indicating the general insufficiency of a one-particle theory; see Ref. 4 and S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row Publishers, Inc., New York, 1961), pp. 63-64. (Schweber’s discussion of the Coulomb field is wrong; for \( E > 0 \) and \( V < 0 \), the probability density is positive everywhere.) However, we have already effectively excluded the far region.


8 See Schweber, Ref. 6. I wish to thank Lars-Erik Lundberg for suggesting this approach.

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Bremsstrahlung Radiation in Plasmas

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A new, short derivation of the radiation formulas from a homogeneous, field free, nonrelativistic plasma is presented. The nonrelativistic assumptions are emphasized.

In this note we present a derivation of the formulas for the bremsstrahlung radiation emitted from a nonrelativistic plasma. These formulas have been previously presented by Dupree and by Birmingham et al. However, their derivations have the disadvantage of being very complicated and lengthy. An alternative method is presented here, using a technique which was previously applied in deriving the relativistically correct emission power. This derivation besides being short has the advantage of clearly showing the necessary nonrelativistic assumptions.

Dawson et al. recently proposed a method of carrying out plasma kinetic theory calculations by expanding the singular distribution function \( f^* \) in terms of the deflections of the particles from
their non-interacting orbits. Thus if
\[ f_\alpha = \langle f_\alpha \rangle + \delta f_\alpha, \]
where \( \langle f_\alpha \rangle \) is the slowly time varying ensemble average and \( \delta f_\alpha \) the fluctuations about \( \langle f_\alpha \rangle \), we directly expand (same notation as Refs. 3 and 4).
\[ \delta f_\alpha = f_\alpha^2 + \delta f_\alpha^2 + \cdots \]
\[ \delta E = E_1 + E_2 + \cdots \]
\[ \delta B_1 = B_1 + B_2 + \cdots \]
Assuming \( \langle E \rangle = \langle B \rangle = 0 \), and \( j_\alpha^s \) isotropic, the second order equations are valid
\[ \left[ (\partial/\partial t) + v \cdot \left( \partial/\partial r \right) \right] R_\alpha^s(q, r, p, t) \]
\[ + q_s E_2 \cdot (\partial f_\alpha^s/\partial p) = 0 \]
\[ \nabla \times E_2 + c^{-1} (\partial B_2/\partial t) = 0 \] \( (1) \)
\[ \nabla \times B_2 = c^{-1} (\partial E_2/\partial t) \]
\[ - (4\pi/c) \sum_\alpha q_s \int d\mathbf{p} R_\alpha^s(q, r, p, t) \mathbf{v} = (4\pi/c) j_s \mathbf{v} \]
\[ \left[ (\partial/\partial t) + v \cdot \left( \partial/\partial r \right) \right] \Psi_\alpha^s(q, r, p, t) \]
\[ = - q_s \left[ E_1 + (v \times B_1/\gamma) \right] \left( \partial \Psi_\alpha^s/\partial \mathbf{p} \right) - \{ \} \] \( (2) \)
\[ j_s \mathbf{v} = \sum_\alpha q_s \int \mathbf{d} \mathbf{p} \Psi_\alpha^s(q, r, p, t) \mathbf{v} \]
where
\[ j_s(q, r, p, t) = \Psi_\alpha^s(q, r, p, t) + R_\alpha^s(q, r, p, t). \]
It can be seen from above that \( \Psi_\alpha^s \) is produced by the direct interaction of the first order fluctuations with the first order fields, and \( R_\alpha^s \) simply represents the shielding of these fluctuations.

The rate at which power is radiated into transverse modes is equal to the rate at which work is done by transverse current fluctuations on the transverse field. Solving Eq. (1) for \( E_2 \) one thus finds for the emission spectrum,
\[ dI/d\Omega = \left[ \left| \frac{1}{2} \right| \left( \Omega^2 - \omega^2 \right)^{1/2} / 8\pi c \right]^2 P(K, \Omega), \] \( (4) \)
where
\[ P(K, \Omega) \equiv \omega^2 + K^2 \gamma^2, \]
\[ (2\pi)^4 j_s^s(q, \omega) = \sum_\gamma \left( \omega^2 / \omega \right) q_s \int d\mathbf{k}' \left( k'^2 / k^2 \right) \int d\omega' n_1^s(k', \omega') n_1^s(k - k', \omega - \omega'). \]

Using this value of \( j_s^s(q, \omega) \) in Eq. (5),
\[ (2\pi)^4 P(k, \omega) = \sum_{\alpha, \beta, \gamma} \omega^2 \omega \mathbf{q} \mathbf{q} \int d\mathbf{k}'' d\omega'' d\mathbf{k}''' d\omega''' \]
\[ \times \exp[i(k - k'') \cdot \mathbf{X} - i(\omega - \omega'') T] \langle k_1'' / k^2 \rangle \langle k_1''' / k'''' \rangle \langle \omega \omega'' \rangle^{-1} \]
\[ \times \langle n_1^s(k - k', \omega - \omega') n_1^s(k', \omega') n_2^s(k'', \omega'') n_1^s(k'' - k''', \omega'' - \omega''') \rangle. \] \( (9) \)
For a homogeneous plasma

\[\langle \eta^{\alpha}(\mathbf{k}_1, \omega_1) \eta^{\beta}(\mathbf{k}_2, \omega_2) \rangle = (2\pi)^{\frac{3}{2}} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{\Omega_{\alpha, \beta, \gamma, \delta}} \sum_{q_\alpha q_\beta q_\gamma q_\delta} \omega_2^2 \omega_1^2 q_\alpha q_\beta \times \frac{\delta(\mathbf{k}_3 - \mathbf{k}_1) \delta(\omega_3 - \omega_1) + \langle \eta^{\alpha}(\mathbf{k}_1, \omega_1) \eta^{\beta}(\mathbf{k}_2, \omega_2) \delta(\mathbf{k}_3 - \mathbf{k}_1) \delta(\omega_3 - \omega_2) \rangle}{\delta(\mathbf{k}_2 - \mathbf{k}_1) \delta(\omega_2 - \omega_1) + \langle \eta^{\alpha}(\mathbf{k}_1, \omega_1) \eta^{\beta}(\mathbf{k}_2, \omega_2) \delta(\mathbf{k}_2 - \mathbf{k}_1) \delta(\omega_2 - \omega_1) \rangle} + O(\varepsilon^2). \quad (10)\]

Using Eq. (10) in Eq. (9), carrying out the integrations and substituting for \(P(K, \Omega)\) in Eq. (4), we find

\[\frac{dI}{d\Omega} = \frac{\Omega}{(2\pi)^{\frac{3}{2}} \Omega_{\alpha, \beta, \gamma, \delta}} \sum_{q_\alpha q_\beta q_\gamma q_\delta} \omega_2^2 \omega_1^2 \left[ \frac{\mathbf{k} \cdot \mathbf{e}^\alpha}{k^3} S^{\alpha\beta}(\mathbf{k}, \omega) S^{\gamma\delta}(\mathbf{K} - \mathbf{k}, \Omega - \omega) \right]
\]

\[+ \frac{(k \cdot e^\alpha)}{k^3} \left[ \frac{S^{\alpha\beta}(\mathbf{k}, \omega) S^{\gamma\delta}(\mathbf{K} - \mathbf{k}, \Omega - \omega)}{k^3} \right] \],

where \(S^{\alpha\beta}(\mathbf{k}, \omega) = \langle \eta^{\alpha}(\mathbf{k}_1, \omega_1) \eta^{\beta}(\mathbf{k}_2, \omega_2) \rangle \) and \(e^\alpha \cdot e^\beta = 2\pi\) is the polarization vector with normalization \(e^\alpha \cdot e^\beta = 2\pi\).

This is the emission formula as given by Tidman and Dupree.\(^7\)

The functions \(S^{\alpha\beta}\) are the first-order density correlations. They are easily calculated using Rostoker's dressed particle model. The first-order particle density perturbation is given by

\[\eta^{\alpha}(\mathbf{k}, \omega) = \eta^{\alpha\sigma}(\mathbf{k}, \omega) \]

\[+ \left[ \frac{L^{\alpha}(\mathbf{k}, \omega)}{\epsilon_1(\mathbf{k}, \omega)} \right] \sum_\gamma (q_\gamma / q_0) n^{\alpha\sigma}(\mathbf{k}, \omega), \quad (11)\]

where \(n^{\alpha\sigma}(\mathbf{k}, \omega)\) is the Fourier transform of the singular particle density \(\delta(\mathbf{x} - \mathbf{x}_m - \mathbf{v}_d t)\) moving in its unperturbed orbit and

\[L^{\alpha}(\mathbf{k}, \omega) = \frac{\omega^2}{k^2} \int d\mathbf{v} \frac{\mathbf{k} \cdot (\partial f^{\alpha\sigma}/\partial \mathbf{v})}{k \cdot \mathbf{v} - \omega} \]

\[\epsilon_1(\mathbf{k}, \omega) = 1 - \sum_\alpha L^{\alpha}(\mathbf{k}, \omega). \]

From the above equations, since \(S^{\alpha\beta}(\mathbf{k}, \omega) = \)

\[\langle \eta^{\alpha}(\mathbf{k}_1, \omega_1) \eta^{\beta}(\mathbf{k}_2, \omega_2) \rangle = \]

\[(2\pi)^{-1} S^{\alpha\beta}(\mathbf{k}, \omega) = \delta(\alpha, \beta) \int d\mathbf{v} f^{\alpha\sigma}(\mathbf{v}) \delta(\omega - k \cdot \mathbf{v}) \]

\[+ \int_{\epsilon_1} L^{\alpha}(\mathbf{k}, \omega) f^{\alpha\sigma}(\mathbf{v}) \delta(\omega - k \cdot \mathbf{v}) \]

\[+ \int_{\epsilon_1} L^{\alpha}(\mathbf{k}, \omega) f^{\alpha\sigma}(\mathbf{v}) \delta(\omega - k \cdot \mathbf{v}) \]

\[+ \int_{\epsilon_1} \int \sum_{q_\alpha q_\beta q_\gamma q_\delta} (q_\alpha q_\beta q_\gamma q_\delta) \int d\mathbf{v} f^{\alpha\sigma}(\mathbf{v}) \delta(\omega - k \cdot \mathbf{v}). \quad (12)\]

In deriving Eq. (12), use made was made of the fact that for a homogeneous plasma

\[\langle \eta^{\alpha}(\mathbf{r}_1, \mathbf{t}_1) \eta^{\beta}(\mathbf{r}_2, \mathbf{t}_2) \rangle = \delta(\alpha, \beta) \int d\mathbf{v} f^{\alpha\sigma}(\mathbf{v}) \delta(\omega - k \cdot \mathbf{v}) \]

\[\delta(\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{v}(\mathbf{t}_1 - \mathbf{t}_2)) \],

which results in

\[\langle \eta^{\alpha}(\mathbf{r}_1, \mathbf{t}_1) \eta^{\beta}(\mathbf{r}_2, \mathbf{t}_2) \rangle = 2\int d\mathbf{v} f^{\alpha\sigma}(\mathbf{v}) \delta(\omega - k \cdot \mathbf{v}). \]

Equations (11) and (12) are the final formulae for the plasma bremsstrahlung radiation in terms of the particle distribution functions.

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\(^11\) Notice that second-order equations are needed since the propagating transverse field first appears to this order.
\(^12\) The ensemble average terms have been neglected since they are low-frequency terms and thus do not contribute to radiation. Also we neglect the transverse field as being relativistically small.