RECONSTRUCTION OF LOW-DIMENSIONAL MAGNETOSPHERIC DYNAMICS BY SINGULAR SPECTRUM ANALYSIS

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Abstract. The low dimensionality of magnetospheric activity indicated by previous phase space reconstructions using the AE and AL data suffer from the limitations of these techniques. In this paper the singular spectrum analysis is used to study the global magnetospheric dynamics using the AE data and it yields a correlation dimension $\simeq 2.5$, thus confirming the low dimensionality published earlier. Further, this technique shows that the global magnetospheric dynamics can be described by 3 variables whose dynamical features are obtained from the AE data.

Introduction

The irregular behavior of the magnetosphere as manifested in the auroral electrojet indices AE and AL has led to the models such as the loading-unloading and driven models of substorms. The irregularity can be due to either the presence of a large number of degrees of freedom in the system or the coupled nonlinear dynamics of a small number of degrees of freedom. A linear prediction filter analysis of the AL index with respect to the product VB_s of the solar wind velocity V and the southward component B_s of the interplanetary magnetic field was performed by Bargatze et al., [1985]. This indicated the presence of two time scales in the magnetospheric activity and also suggested nonlinear behavior. Furthermore, the break in the power spectrum of the AE index [Tsurutani et al., 1990] shows that magnetospheric activity frequency response is distinct from the turbulent solar wind, which has a large number of degrees of freedom.

The possibility that the irregular magnetospheric behavior may be due to low dimensional chaotic dynamics has motivated many recent studies using nonlinear dynamical techniques. The correlation (fractal) dimension, which is a measure of the fractal structure and determines the minimum number of degrees of freedom that describes the system, is one of the dynamical quantities computed using these techniques. For magnetospheric activity, the correlation dimension computed from the AE and AL indices has values in the range 2.2 – 4.2 [Vassiliadis et al., 1990; Roberts, 1991; Roberts et al., 1991; Shan et al., 1991]. However, re-examination of these results using an improved technique proposed by Theiler [1986] shows that the low dimensionality is not sustained in most cases [Prichard and Price, 1992]. This issue is resolved in this paper by using the singular spectrum analysis and it is shown that global magnetospheric dynamics is indeed low dimensional.

The inherent dynamical properties of the magnetosphere obtained from the observational data have important impli-

cations for the solar wind-magnetosphere coupling, which has been interpreted as directly driven [Perrault and Akasofu, 1978] or loading-unloading [McPherron, 1970; Baker et al., 1981] in nature. From the dynamical point of view, the loading-unloading aspects of the substorm activity may be viewed as similar in nature to a dripping faucet which could be chaotic [Baker et al., 1990]. Considering the dominant electrodynamic processes, such models have been further developed to an analogue model based on the dynamics of magnetic flux through a Faraday loop in the magnetotail and this model reproduces many observed dynamical features [Klimas et al., 1992]. On the other hand a linear model that relates the auroral indices to the interplanetary magnetic field and solar wind flow has also given a high correlation coefficient [Goertz, C.K., et al., Prediction of geomagnetic activity, submitted to *Journal of Geophysical Research*, 1991]. In these studies the crucial question has been how many degrees of freedom the system has, or equivalently, how many variables are needed to describe the system. The effective number of variables needed to model magnetospheric dynamics and the dynamical behavior of these variables are determined in this letter by applying the techniques of embedding and singular spectrum analysis to the AE data.

Singular Spectrum Analysis of Time Series Data

The nonlinear dissipative nature of the magnetospheric dynamics suggests the existence of a low-dimensional attractor. An important feature of such a nonlinearly coupled system is that the time series data of one of its variables could contain the essential details of the evolution of all the variables of the system. This is true even though the number of other variables of the system and their time behavior are not directly known, and this has led to the embedding technique of phase space reconstruction [Packard et al., 1980; Takens, 1981]. Given the time series measurements of a variable x(t) from among the many describing the system, the first step in the embedding technique is to construct an m-component delay vector X_i at time $t = t_i$ as $X_i = [x_1(t_i), x_2(t_i), ... x_m(t_i)]$, where $x_k(t_i) = x(t_i + (k-1)\tau)$ and τ is an appropriate time delay large enough to overcome the autocorrelation effect and small enough to resolve the physical processes of interest. In this reconstructed phase space the distribution of state vectors may be studied by defining correlation sums whose scaling behavior leads to the correlation dimension ν of the system [Grassberger and Procaccia, 1983]. This technique may be readily applied when the time delay τ and the embedding dimension m are specified. The time delay τ is often chosen to be close to the autocorrelation time, which for the 1 min averaged AE index data beginning January 1, 1983 [Vassiliadis et al., 1990; Fig.1] is ≈ 100 min. However this value is close to the autocorrelation time the solar wind turbulence and shorter time delays are more appropriate, e.g., the mutual
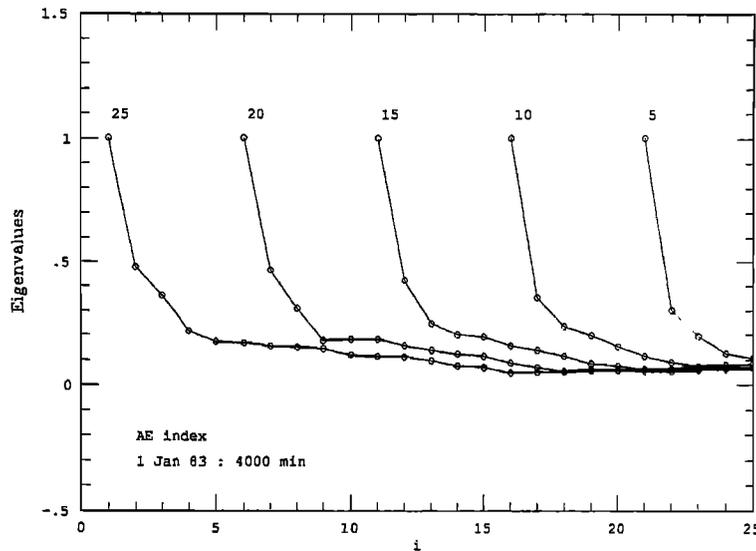
Fig. 1. The normalized singular spectra computed from the first 4000 min of the 1 min averaged AE data beginning January 1, 1983 [Vassiliadis et al., 1990; Fig. 1] with τ = 40 min and m = 5, 10, 15, 20 and 25 as labelled. The eigenvalues beyond the first 3 are close to each other and define the noise floor.

information time ≈ 11 min [Prichard and Price, 1992]. The embedding dimension m should have values $\geq 2\nu + 1$, for a proper embedding [Takens, 1981].

The number of variables actually required to describe the system should be between the nearest integer above the correlation dimension ν and $2\nu+1$ [Takens, 1981]. Thus with the values of ν in the range 2.2–4.2, the magnetospheric activity may be described with 3 to 10 variables. From the modeling point of view it is important to determine the effective number of variables and their dynamical behavior. This can be obtained by using the singular spectrum analysis [Broomhead and King, 1986], also known as the principal component analysis. This technique estimates the effective number of variables of the system from the spectrum of eigenvalues or singular values of a covariance matrix which is defined as follows. From the m-dimensional delay vectors X_i's, a trajectory matrix can be constructed as

$$X = N^{-1/2} \begin{bmatrix} x_1(t_1) & x_2(t_1) & . & . & x_m(t_1) \\ x_1(t_2) & x_2(t_2) & . & . & x_m(t_2) \\ . & . & . & . & . \\ . & . & . & . & . \\ x_1(t_N) & x_2(t_N) & . & . & x_m(t_N) \end{bmatrix},$$

where N is the number of such vectors which is close to the number of data points. This N $\times$ m matrix contains all the dynamical features embodied in the observational data. If the system is low dimensional, its essential dynamics can be described by a smaller number of linearly independent vectors that can be obtained from a singular spectrum analysis of the N $\times$ N matrix XX^T. Further, an eigenvalue analysis shows that the relevant eigenvalues and eigenvectors can be obtained from the m $\times$ m covariance matrix $Z = X^T X$. This is a considerable advantage as the number of vectors N $\approx$ 30,000 and m $\approx$ 10-25 in the present case. In the ideal case the number of eigenvalues of this matrix corresponding to the number of independent variables will be non-zeroes and the rest will be zeroes. However, the inevitable noise in the data makes1 all the eigenvalues finite, with some of them corresponding to the noise level and thus defining a noise
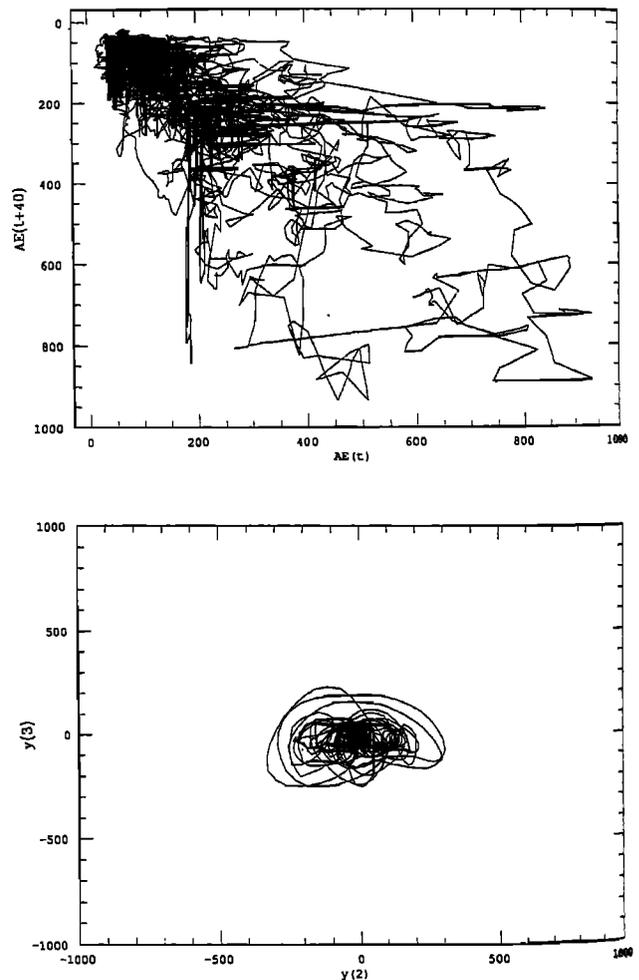

Fig. 2. The phase space of magnetospheric activity reconstructed from the AE data. The upper plot is from a time delay embedding – x(t) versus x(t+τ), with τ = 40 min and the lower one is a plot of the variables y_2 and y_3 obtained by projections along the eigenvectors.
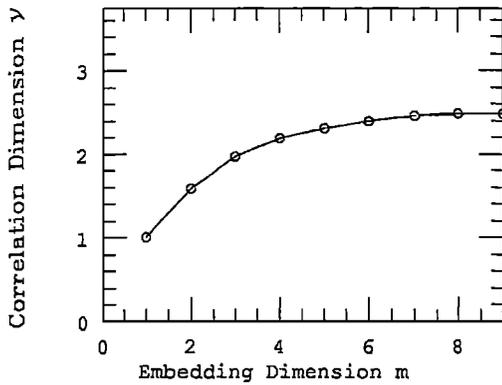
Fig. 3. The computation of the correlation dimension ν using the principal components obtained from the AE data. The function $\ln C(r)/\ln r$ is computed by a least squares fit and saturates for $m \geq 4$ yielding $\nu \simeq 2.5$.

floor. The number of eigenvalues above this noise floor is then an estimate of the number of degrees of freedom. The corresponding eigenvectors give the orthogonal directions in the embedding space along which the dominant dynamical features are represented adequately.

Principal Coordinates of Magnetospheric Activity

The AE and AL data have been widely used to analyze the magnetospheric activity and from these the vectors X_i, the trajectory matrix X and the covariance matrix Z can be readily constructed. The eigenvalues of Z and the corresponding eigenfunctions are then obtained by using the singular value decomposition method. The singular spectrum of 1 min averaged AE index data consisting of 4000 points beginning January 1, 1983 are shown in Figure 1, where the eigenvalues normalized by the first (also the largest) eigenvalue are plotted. The curves correspond to different embedding dimensions as marked and $\tau = 40$ min. The τ values are chosen by considering the loading-unloading time scales of 20-60 min for substorms. The eigenvalues (open circles) corresponding to a particular embedding dimension are joined in sequence in this figure. An important feature of the eigenvalues is the decrease in the separation δ between consecutive values, which is used to define a noise floor within which δ is small. The saturation of the higher eigenvalues beyond the first 3 in Figure 1 is clearer in the case of higher embedding dimensions, e.g., 20 or 25. In the m dimensional embedding space, the eigenvalues yield the relative strengths of the orthogonal principal directions and from Figure 1 it is clear that the directions associated with the first 3 eigenvalues are the dominant ones.

The time series of the principal variables can be readily obtained by projecting the AE time series along the orthogonal directions defined by the eigenvectors in the embedding space. Considering the first 3 eigenvalues, the projections along the corresponding eigenvectors give the 3 principal components (y_1, y_2, y_3). A plot of y_2 versus y_3 is shown in Figure 2 (bottom) along with the reconstructed phase space from the time delay embedding (top). While the reconstruction using the time-delay embedding indicate a low dimensional dynamical system, much cleaner trajectories are obtained from the singular spectrum analysis. The widespread randomness in the time delay reconstruction is thought to be due to the turbulence in the solar wind which drives the mag-

netosphere. On the other hand, the singular spectrum analysis significantly removes the turbulent or random effects and yields much smoother trajectories, as is evident from Figure 2. The magnetospheric response to the solar wind input consists of the directly driven component which is turbulent like the solar wind, and the component due to the internal magnetospheric dynamics. The singular spectrum analysis effectively separates the dynamical component from the noise, thus acting as an effective noise reduction technique. Thus the features shown in Figure 2 are more representative of its internal dynamics. The solar wind-magnetosphere system is a natural input-output system but the output (AE, AL) is dominated by the contributions from the turbulent input (B_s) and it is often difficult to extract the internal dynamical features from the output data alone. From this viewpoint, an input-output approach [Casdagli, 1992] to the analysis of magnetospheric activity has been suggested by Prichard and Price [1992]. Since the time scales of substorms are usually much smaller than the duration for which the IMF remain southward, the magnetospheric activity can be treated as an autonomous system.

Correlation Dimension from the Principal Components

The computation of the correlation dimension using the time delay embedding has a practical difficulty connected with the effect of noise. In this technique, a trajectory describing the evolution of a dynamical system in its phase space is constructed and the correlations between the different parts of it that intersect a region of the phase space are used to compute the correlation dimension [Grassberger and Procaccia, 1983]. However the correlation sum $C(r)$ includes not only the correlations between different passes of the trajectory through a specified region but also those along the trajectory itself. The latter naturally yield no relevant geometrical information and thus need to be excluded, e.g., by introducing a cut off parameter W that determines the number of excluded points along the trajectory [Theiler, 1986]:

$$C(r) = \lim_{N \to \infty} \frac{2}{(N-W)(N-W+1)} \sum_{j=W}^{N} \sum_{i=1}^{N-j} \Theta(r - \mid X_i - X_{i+j} \mid).$$

When this technique is used the values of the correlation dimension computed from many segments of AE and AL data are no longer convergent, except for the AE data of April 1-5, 1983 for which ν converges to a value of 3.4 even with W=100 [Prichard and Price, 1992]. This apparent lack of convergence arises mainly from the fact that the magnetospheric dynamics is driven by the solar wind which is turbulent and strong components of its randomness are present in the AE and AL data.

The variables y_i obtained above from the singular spectrum analysis are practically free from the noise and can be used to compute the correlation dimension by constructing the vector $Y_i = [y_1(t_i), y_2(t_i), ..y_m(t_i)]$. The correlation sum $C(r)$ is readily defined by using these vectors and the correlation dimension $\nu \approx \Delta \log C(r)/\Delta \log r$ computed as usual. The convergence of ν computed with the least squares is shown in Figure 3 and three other techniques, viz. averages, smoothed averages and error minimization by χ^2 test yielded very close values. The "embedding" in this case is the choice of the number of components in the state vector Y_i, e.g., in the m = 3 case the first 3 components are chosen to make

up the vector. From the nature of the singular spectra shown in Figure 1, ν is expected to saturate with m = 4 or 5, and this is obvious in Fig. 3. The values ν computed in this fashion are found to be independent of the spurious autocorrelation effects discussed earlier and the computations with up to W = 100 did not affect the convergence. This remarkable feature is due mainly to the noise reducing features of singular spectrum analysis, as is evident from Figure 2. As the vectors Y_i's are mostly independent of noise, viz. the solar wind turbulence, this value of $\nu \simeq 2.5$ is representative of the internal magnetospheric dynamics. Computations with τ values in the range 20 – 100 min. and larger data sizes (up to 28,800 points) have yielded similar results.

Conclusion

The magnetosphere can be viewed as a dissipative nonlinear dynamical system whose inherent properties can be obtained from the time series AE data. The phase space reconstruction using the singular spectrum analysis has given a distinct internal dynamical behavior that is low dimensional and which can be described by 3 variables; and a fractional dimension $\simeq$ 2.5, confirming earlier published results. The principal coordinates or variables y_i's obtained here contain the dominant features of the magnetospheric dynamics and can be used to construct the dynamical equations. However it should be noted that they may not correspond directly to the physical variables. The current dynamical techniques do not provide a direct way to identify the principal variables with the physical ones and this has to come from other considerations. There are limitations of the techniques of estimating the correlation dimension [Theiler, 1990] and the number of degrees of freedom by the singular spectrum analysis [Palus and Dvorak, 1992]. In this letter a combination of these techniques have been used to used to overcome the autocorrelation effect highlighted by Prichard and Price [1992].

The loading-unloading model of geomagnetic activity [McPherron, 1970; Baker et al., 1981] consists of the loading phase in which the magnetotail field increases due to the day side reconnection, and the unloading phase in which the stored magnetic energy is suddenly released. In the directly driven model geomagnetic activity is directly related to the solar wind and no triggering or sudden release of energy is involved [Perreault and Akasofu, 1978]. From the dynamical point of view these two processes are distinct and the AE index data is analyzed here to isolate the relative features of these models in the observational data. The turbulent solar wind has many degrees of freedom and according to the directly driven model, the substorms as measured by the AE and AL indices should be high dimensional. On the other hand the loading-unloading model may yield a low dimensionality, reflecting internal magnetospheric dynamics. In this sense the dynamical features shown in Figure 2 correspond to the loading-unloading aspects. Further, the magnetosphere has been modeled by a 3 variable RLC circuit [Vassiliadis et al., 1992] which predicts the AL index from the solar wind B_s with correlation coefficient ~80%, thus providing further evidence of its low dimensionality.

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