Effect of magnetic field on propagation of nonlinear ion-acoustic wave

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(Received 1 July 1975; final manuscript received 7 January 1976)

The Korteweg–de Vries equations describing the propagation of slow and fast nonlinear ion-acoustic waves in a magnetic field are derived by using a perturbation method. The effect of the direction of propagation on the amplitude and width of the solitary solutions are discussed.

The propagation of the nonlinear ion-acoustic waves in a collisionless plasma has been studied extensively. In the absence of an external magnetic field, the evolution of a finite amplitude, long wavelength ion-acoustic wave is governed by the Korteweg–de Vries equation[1–4]

$$\frac{\partial \phi}{\partial \tau} + a\phi \frac{\partial \phi}{\partial \xi} + b \frac{\partial^3 \phi}{\partial \xi^3} = 0 .$$

In the present note we study the effect of an external magnetic field B_0 on the propagation of ion-acoustic waves in a collisionless plasma consisting of a cold ion fluid and hot and isothermal electrons. At low frequencies a plasma with $B_0 \neq 0$ can support two ion-acoustic modes with frequencies[5–7]

$$\omega_{1,2}^2 = \tfrac{1}{2}\{\omega_s^2 + \omega_{Bi}^2 \pm [(\omega_s^2 + \omega_{Bi}^2)^2 - 4\omega_s^2\omega_{Bi}^2 \cos^2\theta]^{1/2}\}, \quad (1)$$

where $\omega_s = kV_s(1 + k^2\lambda_D^2)^{-1/2}$ is the ion-acoustic frequency in the absence of the magnetic field, $V_s = (T_e/m_i)^{1/2}$ is the ion-acoustic velocity, ω_{Bi} is the ion-cyclotron frequency, θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$, and $\lambda_D = (T_e/4\pi n_0 e^2)^{1/2}$ is the electron Debye length. $\mathbf{B}_0$ is along the z axis and $\mathbf{k}$ is in the $x-z$ plane. Equation (1) is valid under the conditions

$$\omega \ll k_\parallel T_e , \quad k^2\rho_{e,i}^2 \ll 1 , \quad \text{and} \ |\omega - n\omega_{Bi}| \gg k_\parallel V_{Ti} , \quad (n = 0, \pm 1)$$

where $V_{T\alpha} = (T_\alpha/m_\alpha)^{1/2}$ is the thermal velocity, $\rho_\alpha = V_{T\alpha}/\omega_{B\alpha}$ is the electron Larmor radius for $\alpha = e$ and the ion Larmor radius for $\alpha = i$, and $k_\parallel = k\cos\theta$.

If $k^2V_s^2 \gg \omega_{Bi}^2$ and $k^2\lambda_D^2 \ll 1$, Eq. (1) gives the fast ion-acoustic wave

$$\omega_1 = kV_s\left(1 + \frac{\sin^2\theta}{2k^2R^2} - \frac{k^2\lambda_D^2}{2}\right), \quad (2)$$

where $R = V_s/\omega_{Bi}$. The effect of the magnetic field thus is small, since $k^2R^2 \gg 1$. The plasma must be dense and the magnetic field weak to support this mode because $\omega_{pi}^2 = V_s^2/\lambda_D^2 \gg \omega_{Bi}^2$.

If $k^2V_s^2 \ll \omega_{Bi}^2$ and $k^2\lambda_D^2 \ll 1$, Eq. (1) yields the slow ion-acoustic mode

$$\omega_2 \simeq k_z V_s(1 - k_\perp^2R^2/2 - k^2\lambda_D^2/2), \quad (3)$$

where $k_\perp = k\sin\theta$. This mode is supported by a rare plasma with strong magnetic field present. The effect of magnetic field is small as in the fast mode because $k^2R^2 \ll 1$ in this case.

To simplify the analysis we shall define a new axis ξ in the $x=z$ plane and at an angle θ with the z axis. We shall then consider the one-dimensional waves propagating along the ξ axis. The system is completely described by the equation of continuity, the equation of motion, and the Poisson equation[6]

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \xi}(nu_\xi) = 0 , \quad (4)$$

$$\frac{\partial u_x}{\partial t} + u_\xi \frac{\partial u_x}{\partial \xi} = \frac{1}{\epsilon^{1/2}\mu\sigma_\nu} u_y - \frac{\sin\theta}{\sigma_\nu^2}\frac{\partial \phi}{\partial \xi} , \quad (5)$$

$$\frac{\partial u_y}{\partial t} + u_\xi \frac{\partial u_y}{\partial \xi} = -\frac{1}{\epsilon^{1/2}\mu\sigma_\nu} u_x , \quad (6)$$

$$\frac{\partial u_z}{\partial t} + u_\xi \frac{\partial u_z}{\partial \xi} = -\frac{\cos\theta}{\sigma_\nu^2}\frac{\partial \phi}{\partial \xi} , \quad (7)$$

$$\epsilon \frac{\partial^2 \phi}{\partial \xi^2} = n_e - n , \quad (8)$$

and

$$n_e = \exp\phi , \quad u_\xi = u_x \sin\theta + u_z\cos\theta .$$

Here, e is the ion charge, and the subscript $\nu = 1$ corresponds to the fast wave, $\sigma_1 = \cos\theta$, and $\nu = 2$ to the slow wave, $\sigma_2 = 1$. The small parameters ϵ and μ are defined as

$$\epsilon = \lambda_D^2/l^2 \ \text{and} \ \mu = \omega_{Pi}/\omega_{Bi} ,$$

where l is the characteristic dimension of the inhomogeneity. $\epsilon \ll 1$ is the same for both the fast and slow modes. However, as discussed earlier, $\mu \gg 1$ for the fast mode and $\mu \ll 1$ for the slow mode. In Eqs. (4) to (8), the ion density n and the electron density n_e have been normalized to the background density n_0, the velocity of the cold ions $u_i (i = x, y, z)$ to $\sigma_\nu V_s$, the electrostatic potential ϕ to the characteristic potential (T_e/e), the distance ξ to l, and the time t to $(\sigma_\nu V_s/l)$.

We introduce the new variables

$$\zeta = \xi - \lambda_\nu t , \quad \tau = \epsilon t , \quad (9)$$

where the velocity λ_ν is to be determined later. Also, we expand n, u_i, and ϕ in power series in ϵ:

$$n = 1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \cdots ,$$
$$u_i = \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \cdots , \quad (10)$$
$$\phi = \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \cdots .$$

For the fast ion-acoustic wave $\mu \gg 1$, so we choose $\mu = b\epsilon^{-3/2}$ (b = const.) and also $\sigma_1 = \cos\theta$. On using these values in Eqs. (4)–(8) and substituting from Eqs. (9)

and (10) we get equations of different orders in ϵ. The ϵ order equation yields

$$u_y^{(1)}=0, \quad \phi^{(1)}=n^{(1)}, \quad u_\xi^{(1)}=\lambda_1\phi^{(1)}, \quad \lambda_1=1/\cos\theta . \quad (11)$$

Using these in the ϵ^2 equations of (4), (5), (7), and (8), $u_\xi^{(2)}$, $\phi^{(2)}$, and $n^{(2)}$ may be eliminated to get the Korteweg-deVries equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\lambda_1\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+\tfrac{1}{2}\lambda_1\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0\left(\lambda_1=\frac{1}{\cos\theta}\right). \quad (12)$$

For the slow ion-acoustic wave $\mu\ll 1$, so we choose $\mu=b\epsilon^{1/2}$ and also $\sigma_2=1$. Then, the same analysis is carried out as before. The lowest order (ϵ^0) equations give

$$u_x^{(1)}=u_y^{(1)}=0 . \quad (13)$$

The ϵ order equations yield

$$u_x^{(2)}=0, \quad \phi^{(1)}=n^{(1)}, \quad u_\xi^{(1)}=u_z^{(1)}\cos\theta, \quad \lambda_2=\cos\theta . \quad (14)$$

From the ϵ^2 equations of (4), (7), and (8), and these results, we get the Korteweg–de Vries equation for the slow ion-acoustic wave

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\lambda_2\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+\tfrac{1}{2}\lambda_2\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0 \quad (\lambda_2=\cos\theta). \quad (15)$$

Thus, the propagation of small but finite amplitude ion-acoustic waves in an external magnetic field, both the fast and slow modes, are governed by the Korteweg–de Vries equation. The forms of this equation for two modes, Eqs. (12) and (15), are very similar, the only difference being in the values of the phase velocities. Also, the phase velocity of one mode is the inverse of the phase velocity of the other. The dispersion length, defined as the ratio of the coefficient of the dispersive term to the phase velocity is independent of θ and has the value 0.5 for both modes.

By transforming to the moving frame $\eta=\xi-U\tau$, and using the boundary conditions

$$\phi^{(1)}, \quad \frac{d\phi^{(1)}}{d\eta}, \quad \frac{d^2\phi^{(1)}}{d\eta^2}\to 0 \text{ as } \eta\to\pm\infty ,$$

Eqs. (12) and (15) may be integrated to yield localized stationary solutions

$$\phi^{(1)}(\eta)=(3U/\lambda_\nu)\,\mathrm{sech}^2[(U/2\lambda_\nu)^{1/2}\eta] . \quad (16)$$

This is the well known solitary wave or soliton. The amplitude of the soliton is $(3U/\lambda_\nu)$ and the width $(U/2\lambda_\nu)^{1/2}$ and it propagates to the right with velocity U. For the fast ion-acoustic wave, $\lambda_1=1/\cos\theta$ and hence both the amplitude and the width are directly proportional to $\cos\theta$. In the case of the slow ion-acoustic wave $\lambda_2=\cos\theta$ both the amplitude and the width are inversely proportional to $\cos\theta$. Also, the solution (16) is a solitary wave only when $-\pi/2<\theta<\pi/2$.

Pokroev and Stepanov[7] have studied the propagation of the nonlinear ion-acoustic waves in a magnetoplasma by transforming Eqs. (4) to (8) in terms of Lagrangian variables. They obtain the Korteweg–de Vries equation for the slow wave, but for the fast wave they get a different equation. Here, we obtain the Korteweg–de Vries equation for the fast wave by choosing $\mu=b\epsilon^{-3/2}$. The reason for the disagreement is the different approximations used. For example, the term $k^2\lambda_D^2$ term in Eq. (2) has been retained in the present analysis whereas in Ref. 7 this term has been neglected. This term, as is clear from the structure of Eqs. (12) and (15), is an essential term for obtaining the Korteweg–de Vries equation.

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Nonlinear virtual ion modes

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(Received 30 June 1975; final manuscript received 2 December 1975)

The nonlinear growth and saturation of the nonresonant beat oscillation driven by two ion-acoustic waves have been observed. After saturation, the beat signal relaxes into a resonant ion-acoustic wave.

Two propagating plasma waves (ω_1, k_1) and (ω_2, k_2) may beat together to generate a nonlinear forced oscillation ($\Delta\omega$, Δk), which can interact with a given class of particles of velocity v, such that $\Delta\omega=v\Delta k$. When this forced oscillation does not satisfy the dispersion relation [$\mathcal{E}(\Delta\omega, \Delta k)=0$], the interaction thus produced between the two original waves is known as nonlinear Landau damping.[1] Nonlinear Landau damping of ion acoustic waves has been reported by Ikezi and Kiwamoto,[2] who observed the energy transfer between two driving waves, as well as an interference effect between the beat signal and a linearly propagating low-frequency wave. In the present note, we wish to report observations of the nonlinear growth of the nonresonant low-frequency driven signal.

The experiments were performed in a highly uniform plasma, 1% ionized, 35 cm long and 30 cm diam, pro-