Lecture 13
AC CIRCUITS
CHAPTER 36

PHYSICS 270

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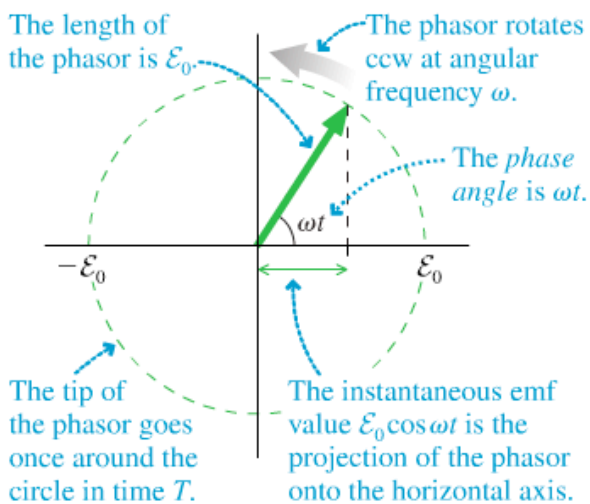
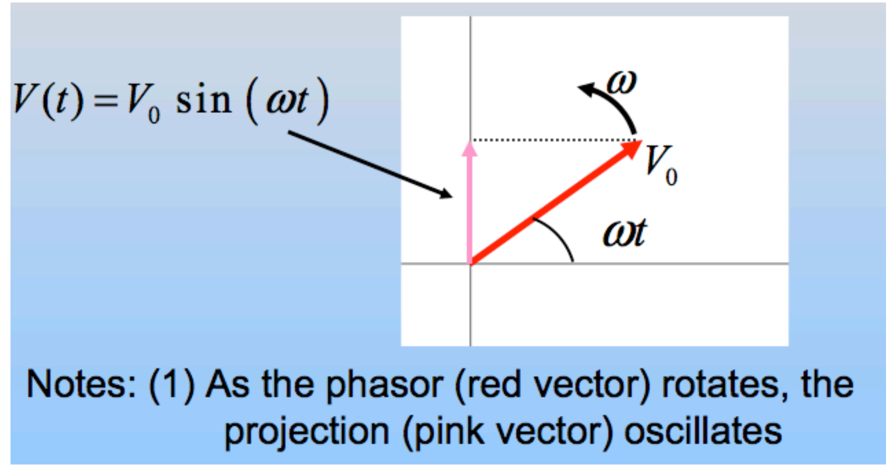
Chapter 36. AC Circuits

Topics:

- AC Sources and Phasors
- Capacitor Circuits (Capacitive Reactance)
- *RC* Filter Circuits
- Inductor Circuits (Inductive Reactance)
- The Series *RLC* Circuit

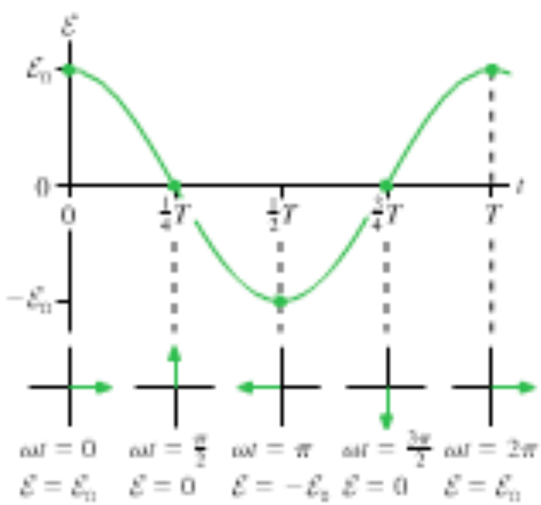
Cos vs. Sin Source





$$T = 2\pi / \omega$$

FIGURE 36.2 The correspondence between a phasor and points on a graph.



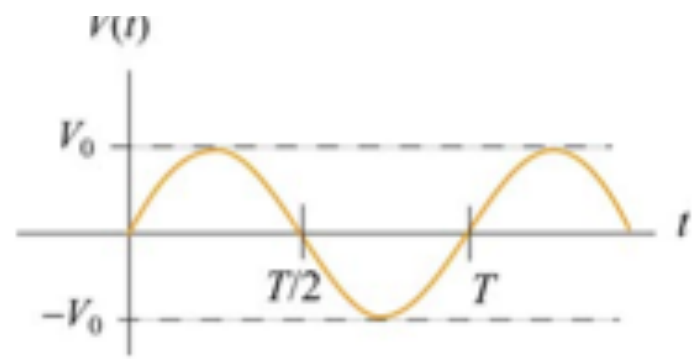


Figure 12.1.1 Sinusoidal voltage source

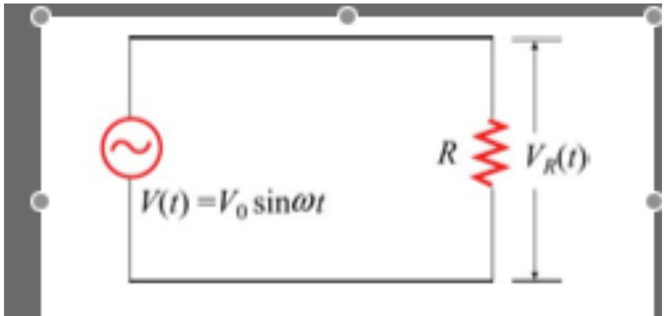
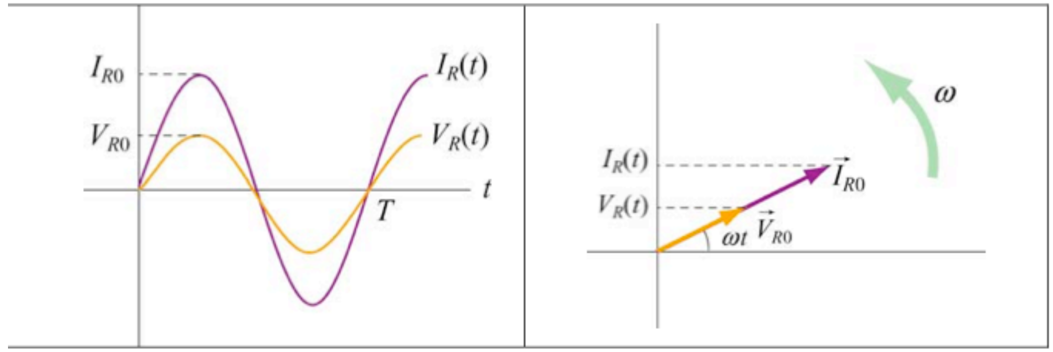


Figure 12.2.1 A purely resistive circuit



$$V_R(t) = I_R(t)R$$

$$V_o \sin(\omega t) - I_R(t)R = 0$$

$$I_R(t) = \frac{V_o}{R}\sin(\omega t)$$

Voltage and current in phase

Issue of Phase

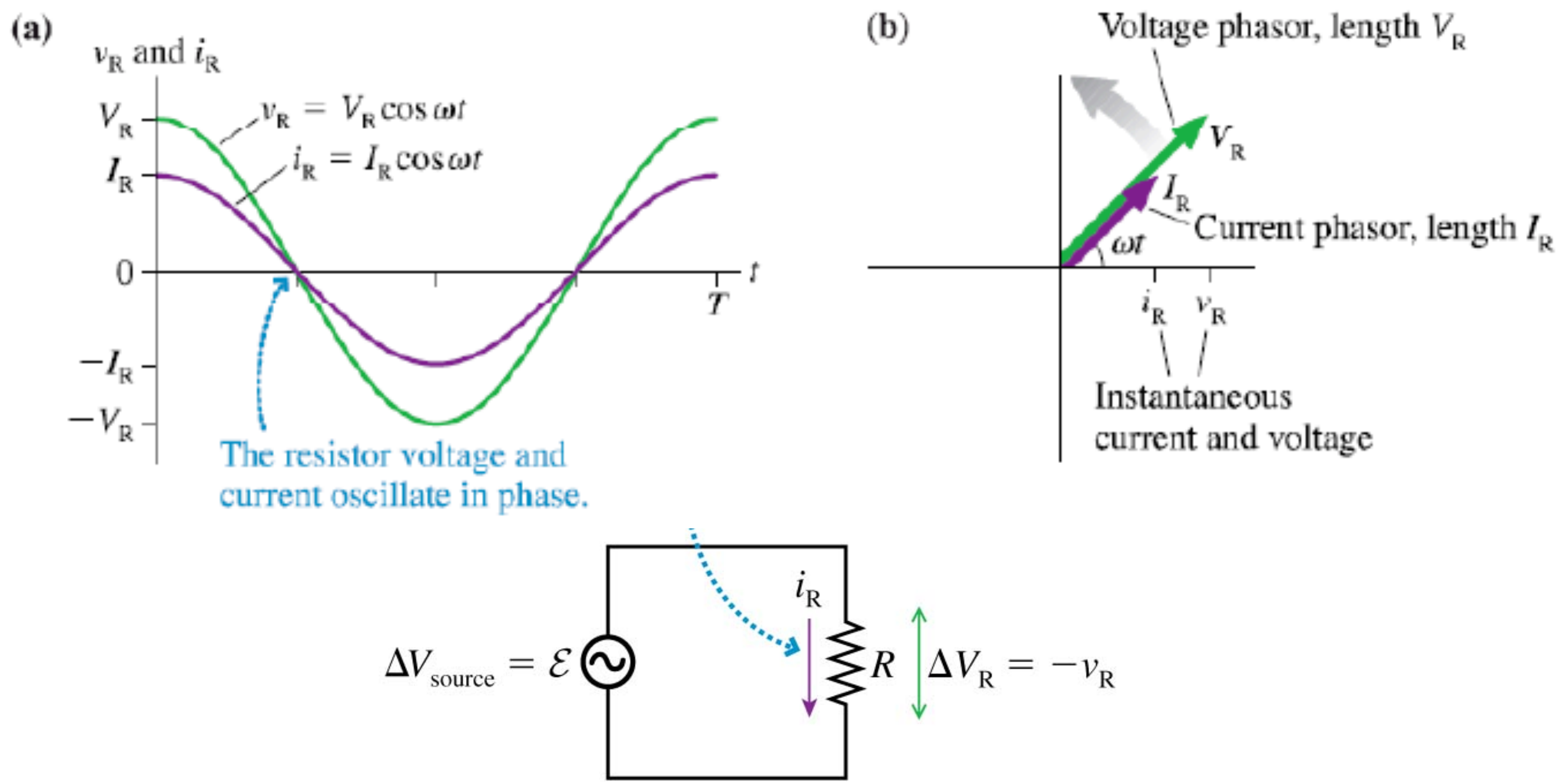
If I take emf as $V_o \sin(\omega t)$ and I find $I(t) = (V_o / R)\sin(\omega t + \phi)$ current leads the voltage by ϕ.

If I take emf as $V_o \cos(\omega t)$ and I find $I(t) = (V_o / R)\cos(\omega t + \phi)$ current leads the voltage by ϕ.

If ϕ is $\pi/2$ it leads by $\pi/2$ if ϕ is $-\pi/2$ it lags by $\pi/2$

AC Circuits - Resistors

FIGURE 36.5 Graph and phasor diagram of the resistor current and voltage. The current and voltage are in phase.



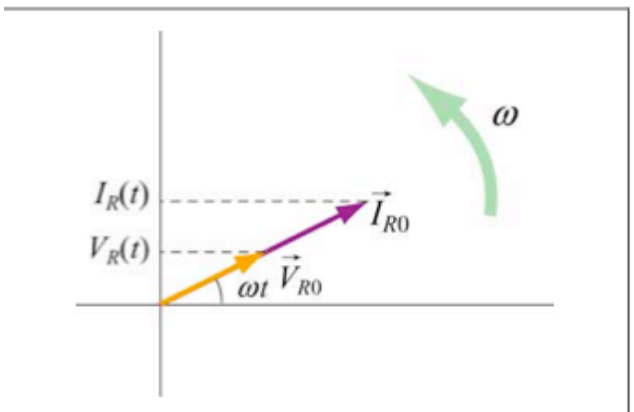
The average value of current over one period can be obtained as:

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)dt = \frac{1}{T}\int_0^T I_{R0}\sin\omega t\ dt = \frac{I_{R0}}{T}\int_0^T \sin\frac{2\pi t}{T}\ dt = 0 \qquad (12.2.3)$$

This average vanishes because

$$\langle \sin\omega t\rangle = \frac{1}{T}\int_0^T \sin\omega t\ dt = 0 \qquad (12.2.4)$$

Similarly, one may find the following relations useful when averaging over one period:



$$\langle \cos\omega t\rangle = \frac{1}{T}\int_0^T \cos\omega t\ dt = 0$$

$$\langle \sin\omega t\cos\omega t\rangle = \frac{1}{T}\int_0^T \sin\omega t\cos\omega t\ dt = 0$$

$$\langle \sin^2\omega t\rangle = \frac{1}{T}\int_0^T \sin^2\omega t\ dt = \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2}$$

$$\langle \cos^2\omega t\rangle = \frac{1}{T}\int_0^T \cos^2\omega t\ dt = \frac{1}{T}\int_0^T \cos^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2}$$

Whether we take sin or cos is a matter of choice. Result the same. $< >=0$, $<{}^2>=1/2$

From the above, we see that the average of the square of the current is non-vanishing:

$$\langle I_R^2(t)\rangle = \frac{1}{T}\int_0^T I_R^2(t)dt = \frac{1}{2}I_{R0}^2$$

It is convenient to define the root-mean-square (rms) current as

$$I_{\text{rms}} = \sqrt{\langle I_R^2(t)\rangle} = \frac{I_{R0}}{\sqrt{2}} \tag{12.2.7}$$

In a similar manner, the rms voltage can be defined as

$$V_{\text{rms}} = \sqrt{\langle V_R^2(t)\rangle} = \frac{V_{R0}}{\sqrt{2}} \tag{12.2.8}$$

The rms voltage supplied to the domestic wall outlets in the United States is $V_{\text{rms}} = 120\ \text{V}$ at a frequency $f = 60\ \text{Hz}$.

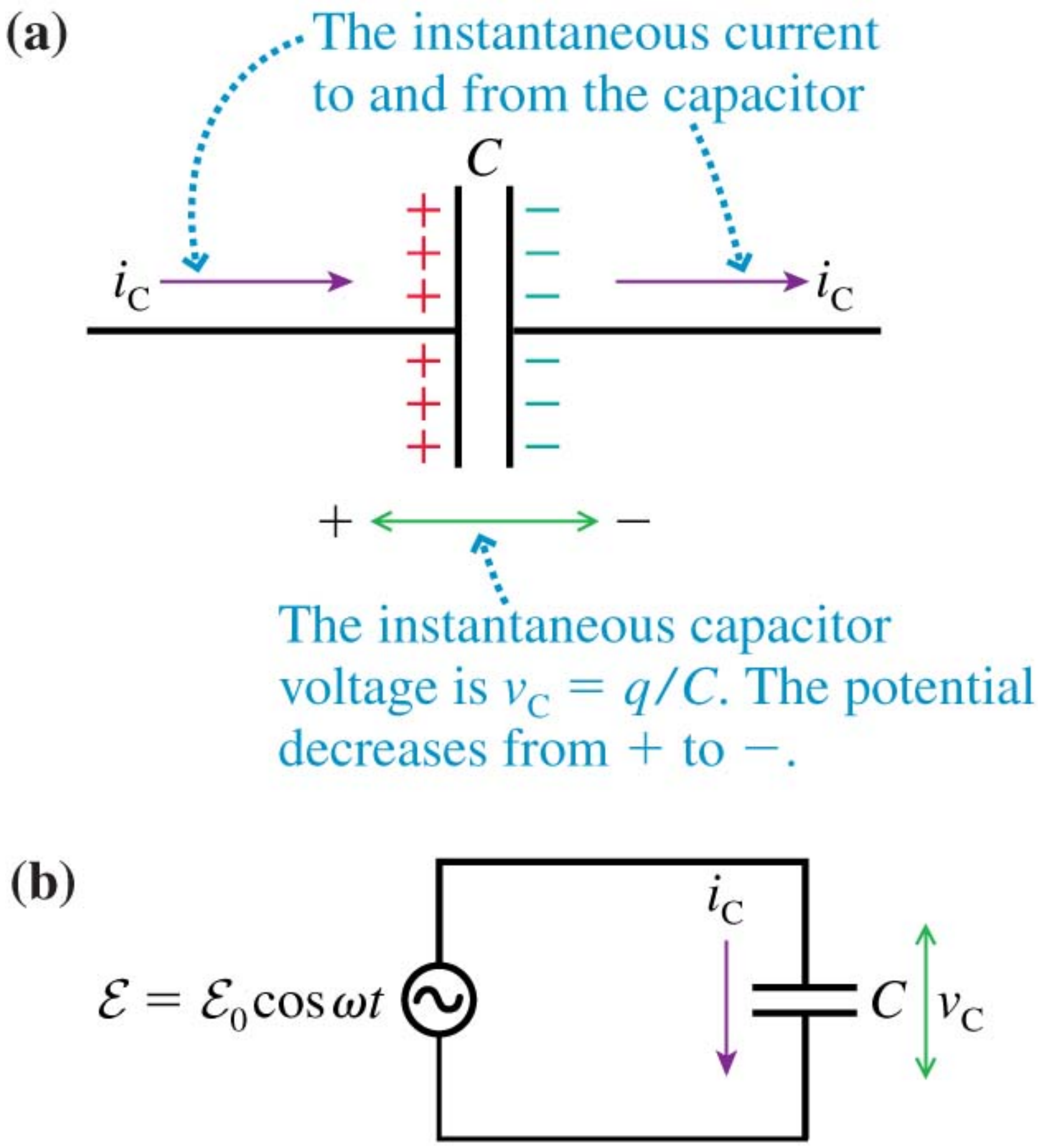
The power dissipated in the resistor is

$$P_R(t) = I_R(t)V_R(t) = I_R^2(t)R \tag{12.2.9}$$

from which the average over one period is obtained as:

$$\langle P_R(t)\rangle = \langle I_R^2(t)R\rangle = \frac{1}{2}I_{R0}^2R = I_{\text{rms}}^2R = I_{\text{rms}}V_{\text{rms}} = \frac{V_{\text{rms}}^2}{R} \tag{12.2.10}$$

FIGURE 36.7 An AC capacitor circuit.



$$v_c = V_c \cos(\omega t)$$

$$q = C v_c = C V_c \cos(\omega t)$$

$$i_c = \frac{dq}{dt} = -\omega C V_c \sin(\omega t)$$

$$i_c = \omega C V_c \cos(\omega t + \frac{\pi}{2}) \equiv$$

$$\equiv I_C \cos(\omega t + \frac{\pi}{2})$$

$$I_C \equiv V_C / X_C$$

$$X_C \equiv 1/\omega C$$

$$v_c = V_c \cos(\omega t)$$

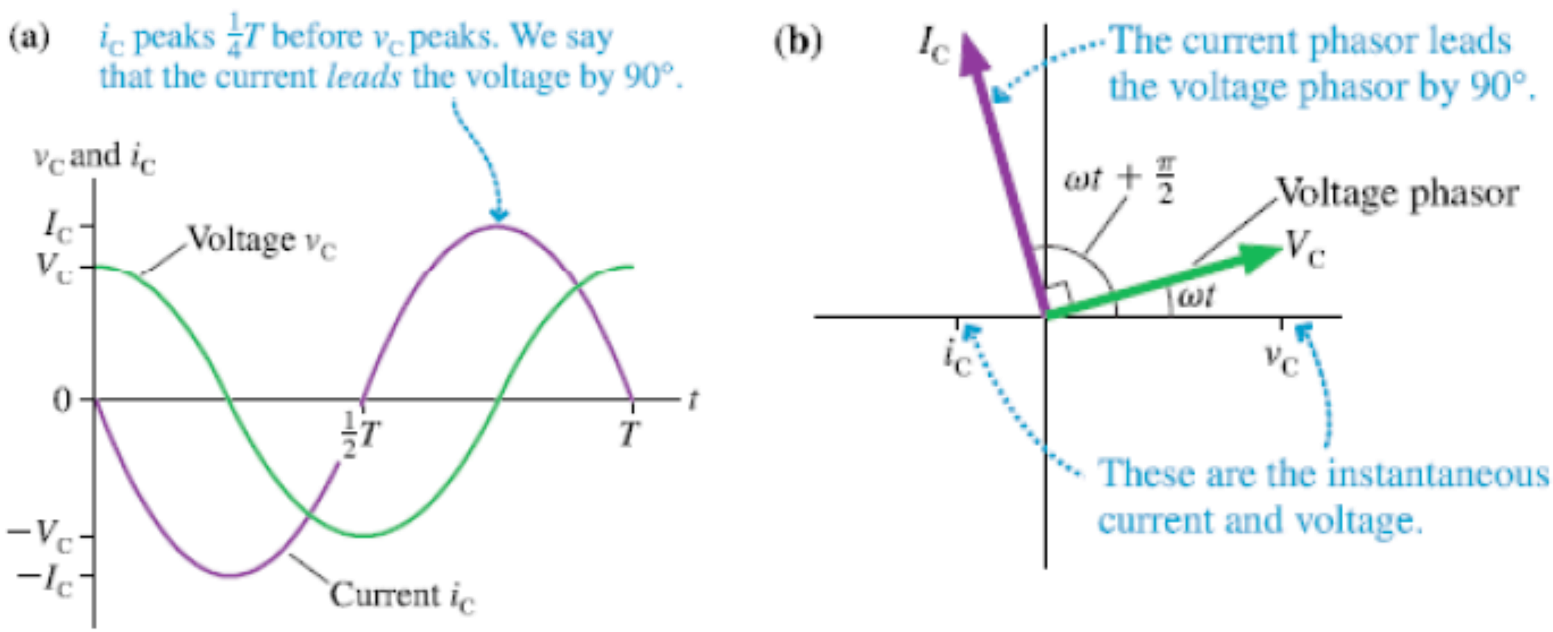
$$i_c = I_C \cos(\omega t + \frac{\pi}{2})$$

$$I_c \equiv \omega C V_c$$

$$I_R = V_R / R$$

$$I_C = V_c / X_C$$

$$X_C \equiv 1/\omega C$$

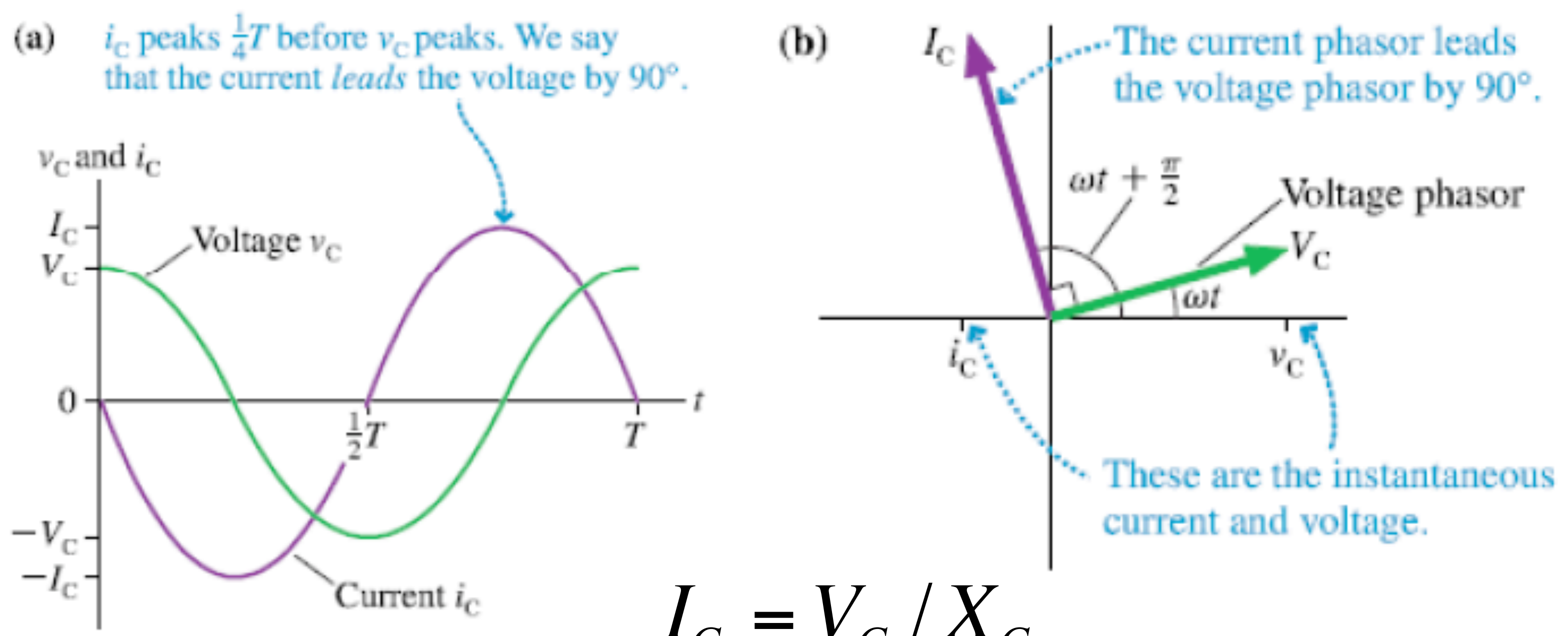


Lead if in the cycle peak occurs first or is ahead in the phasor ccw rotation

Capacitive Reactance - ohms

AC Circuits - Capacitors

ASSESS Using reactance is just like using Ohm's law, but don't forget it applies to only the *peak* current and voltage, not the instantaneous values.



$$I_C = V_C / X_C$$

$$X_C = 1/\omega C$$

Capacitor Circuits

The instantaneous voltage across a single capacitor in a basic capacitor circuit is equal to the instantaneous emf:

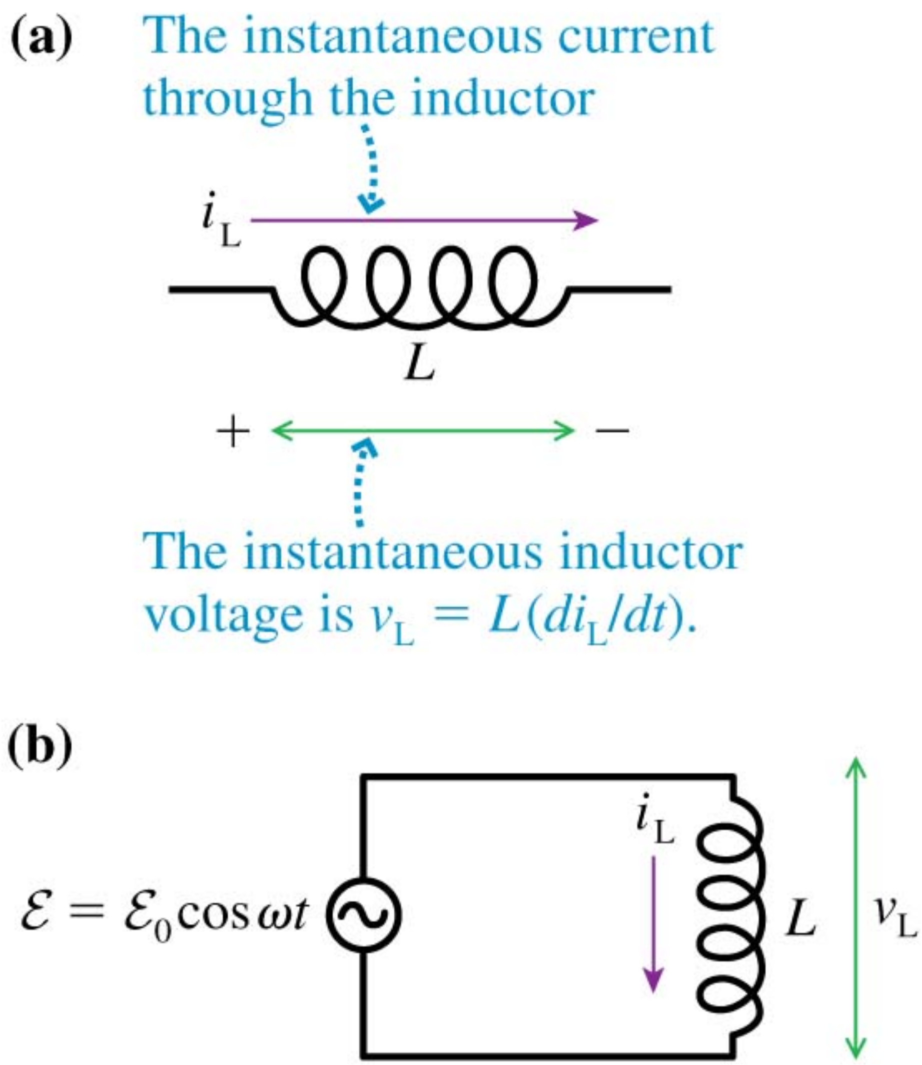
$$v_C = V_C \cos \omega t$$

Where V_C is the maximum voltage across the capacitor, also equal to the maximum emf. The instantaneous current in the circuit is

$$i_C = \omega C V_C \cos\left(\omega t + \frac{\pi}{2}\right)$$

The AC current to and from a capacitor *leads* the capacitor voltage by π/2 rad, or 90°.

FIGURE 36.14 Using an inductor in an AC circuit.

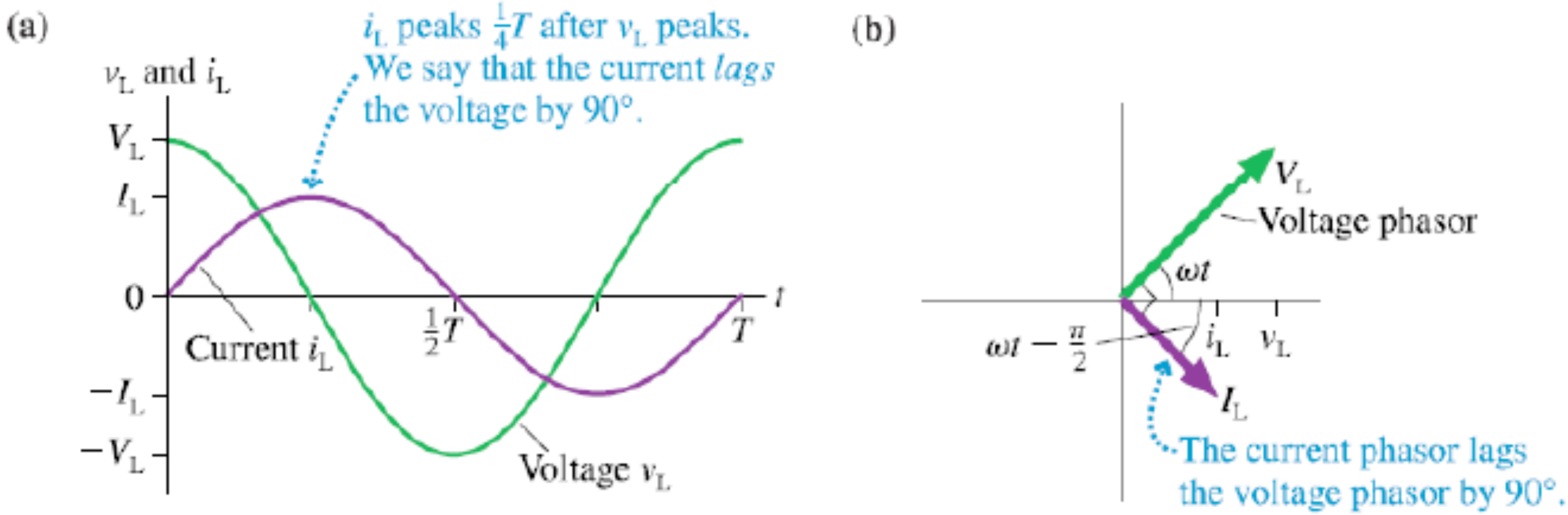


$$v_L = V_L \cos(\omega t) = L(di_L / dt)$$

$$i_L = \frac{V_L}{L}\int \cos\omega t\, dt = \frac{V_L}{\omega L}\sin\omega t = \frac{V_L}{\omega L}\cos\left(\omega t - \frac{\pi}{2}\right)$$
$$= I_L \cos\left(\omega t - \frac{\pi}{2}\right)$$

Equivalent Ohm's Law

$$I_L = V_L / X_L$$
$$X_L \equiv \omega L$$



$$i_L = I_L \cos(\omega t - \pi / 2)$$

$$v_L = V_L \cos(\omega t)$$

Inductor Circuits

The instantaneous voltage across a single inductor in a basic inductive circuit is equal to the instantaneous emf:

$$v_L = V_L \cos \omega t$$

Where V_L is the maximum voltage across the inductor, also equal to the maximum emf. The instantaneous inductor current is

$$i_L = \frac{V_L}{L} \int \cos \omega t \, dt = \frac{V_L}{\omega L} \sin \omega t = \frac{V_L}{\omega L} \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$= I_L \cos\left(\omega t - \frac{\pi}{2}\right)$$

The AC current through an inductor *lags* the inductor voltage by π/2 rad, or 90°.

Inductive Reactance

The inductive reactance X_L is defined as

$$X_L \equiv \omega L$$

Reactance relates the peak voltage V_L and current I_L:

$$I_L = \frac{V_L}{X_L} \quad \text{or} \quad V_L = I_L X_L$$

NOTE: Reactance differs from resistance in that it does *not* relate the instantaneous inductor voltage and current because they are out of phase. That is, $v_L \neq i_L X_L$.

FIGURE 36.10 The capacitive reactance as a function of frequency.

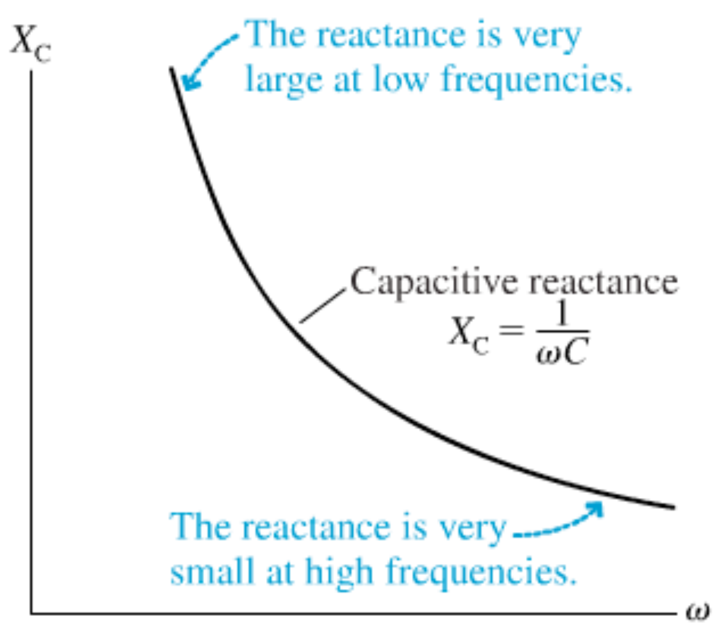
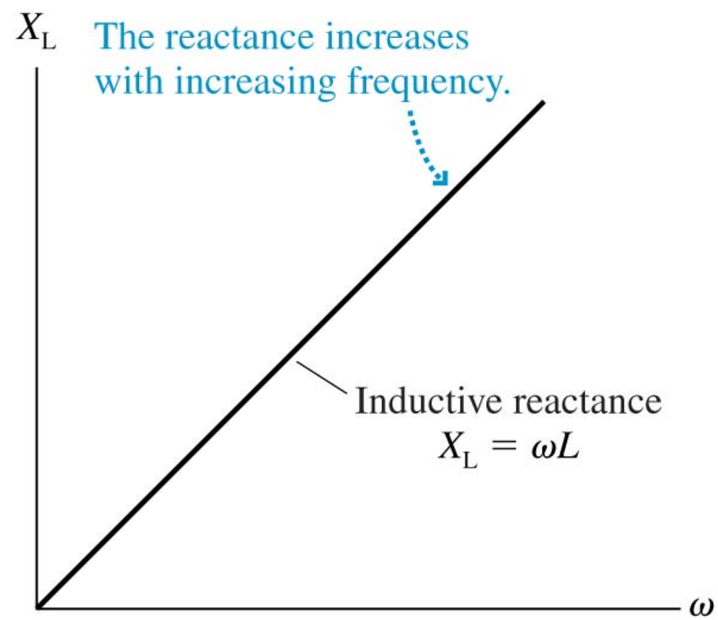


FIGURE 36.15 The inductive reactance as a function of frequency.



AC Circuits: Summary

Element	I_0	Current vs. Voltage	Resistance Reactance Impedance
Resistor	$\frac{V_{0R}}{R}$	In Phase	$R = R$
Capacitor	$\omega C V_{0C}$	Leads	$X_C = \frac{1}{\omega C}$
Inductor	$\frac{V_{0L}}{\omega L}$	Lags	$X_L = \omega L$

Although derived from single element circuits, these relationships hold generally!

Basic circuit elements

Element	i and v	Resistance/ reactance	I and V	Power
Resistor	In phase	R is fixed	$V = IR$	$V_{rms}I_{rms}$
Capacitor	i leads v by 90°	$X_C = 1/\omega C$	$V = IX_C$	0
Inductor	i lags v by 90°	$X_L = \omega L$	$V = IX_L$	0

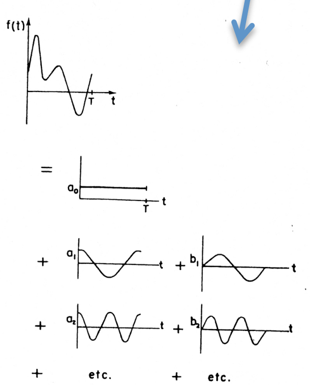
Root mean square quantities $V_{rms}=V/\sqrt{2}$

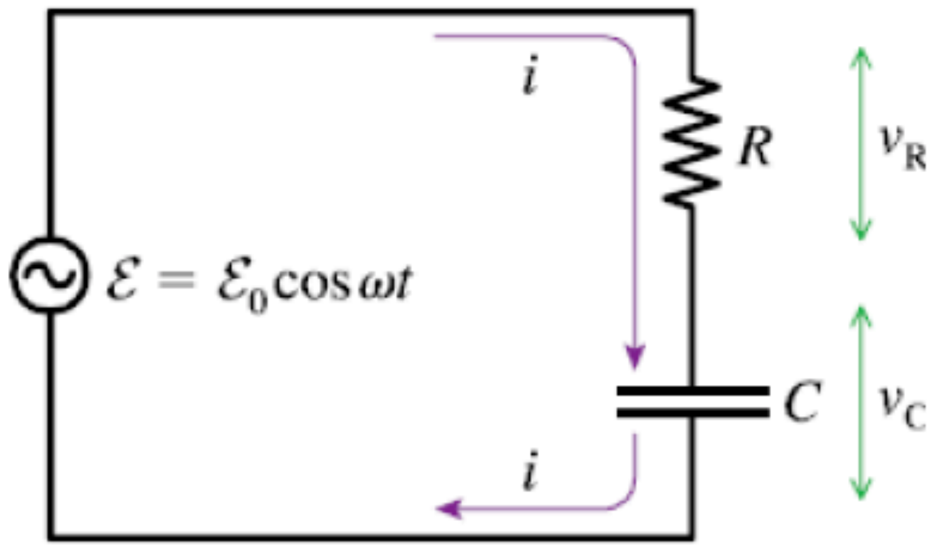
RC Filters – The concept (Fourier analysis)

Any waveform (like voltages driving your speaker when you play music) is a sum of many sinusoidal waveforms of different amplitudes and frequencies.

The ac voltage generator depicted below for an RC circuit is idealized as ONE input frequency, but in general could be a sum of MANY waveforms (like music) with many frequencies.

Goal: Analyze the individual voltages across the resistor and capacitor when an input waveform with any frequency ω and voltage amplitude ε0 is applied across both.





Musical tone characteristics: Loudness (Amplitude of pressure), pitch-period of time for one repetition of the basic pressure function (low pitch long period low frequency, high pitch short period, high frequency)

RC Filters – Analysis

Analyzing an *RC* circuit

1. current is the same at all points in circuit at all time. Choose an arbitrary current vector (time).
2. Phase: V_R in phase with I, I in capacitor leads Vc
   Amplitude: For a given I peak value, we know V_R and Vc peak
3. At any instant in time, we have (Kirchoff's loop law):

$$\vec{V}_R \cdot \hat{e}_x + \vec{V}_c \cdot \hat{e}_x = \vec{\varepsilon} \cdot \hat{e}_x = \varepsilon_o \cos(\omega t)$$

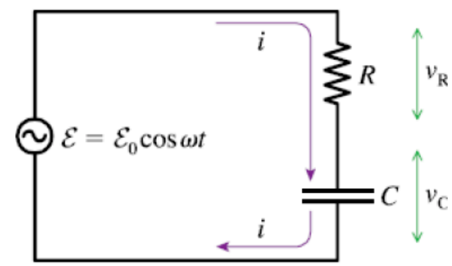
$$\vec{V}_R + \vec{V}_c = \vec{\varepsilon}$$

$$\varepsilon_o^2 = I^2(R^2 + X_C^2)$$

$$I = \frac{\varepsilon_o}{\sqrt{R^2 + +X_C^2}}$$

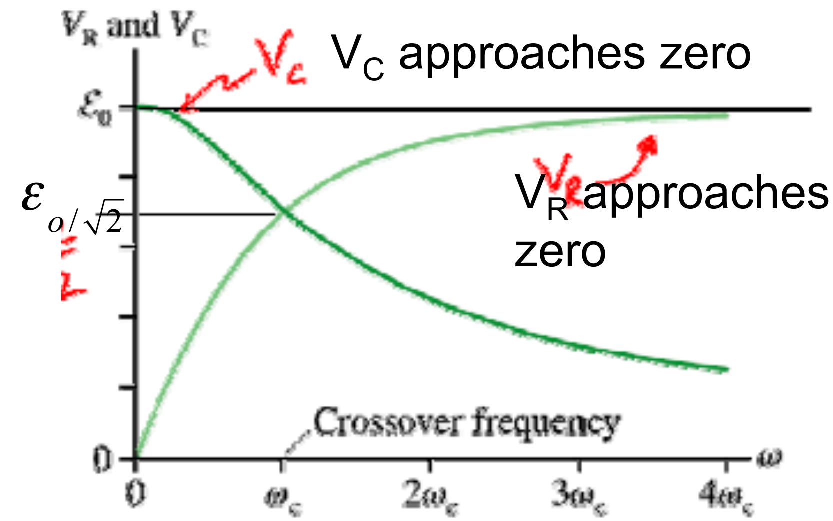
$$V_R = IR$$

$$V_C = IX_C$$



$$V_R = IR = \frac{\mathcal{E}_0 R}{\sqrt{R^2 + X_C^2}} = \frac{\mathcal{E}_0 R}{\sqrt{R^2 + 1/\omega^2 C^2}}$$

$$V_C = IX_C = \frac{\mathcal{E}_0 X_C}{\sqrt{R^2 + X_C^2}} = \frac{\mathcal{E}_0/\omega C}{\sqrt{R^2 + 1/\omega^2 C^2}}$$

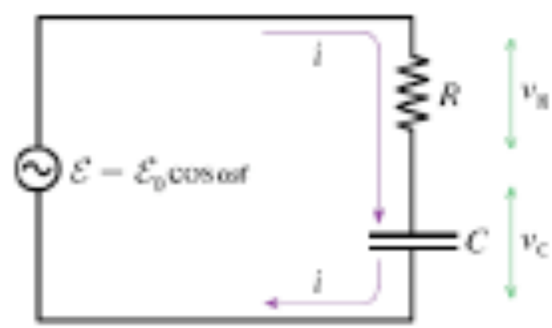


For $\omega \to 0$, ($X_C >> R$), $V_R \to 0$, $V_C \to \varepsilon_o$; like shorted resistor

For $\omega \to \infty$, ($X_C << R$), $V_R \to \varepsilon_o$, $V_C \to 0$; like shorted capacitor

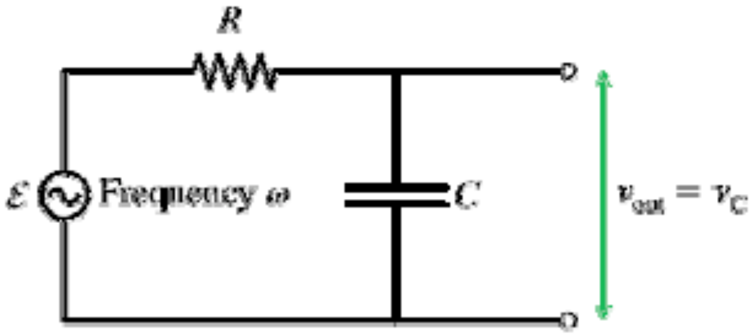
For $\omega = 1/RC$ ($R = X_C$), $V_R = V_C = (1/\sqrt{2})\varepsilon_o = (1/\sqrt{2})\varepsilon_o$

Define cross-over frequency $\omega_c = 1/RC$

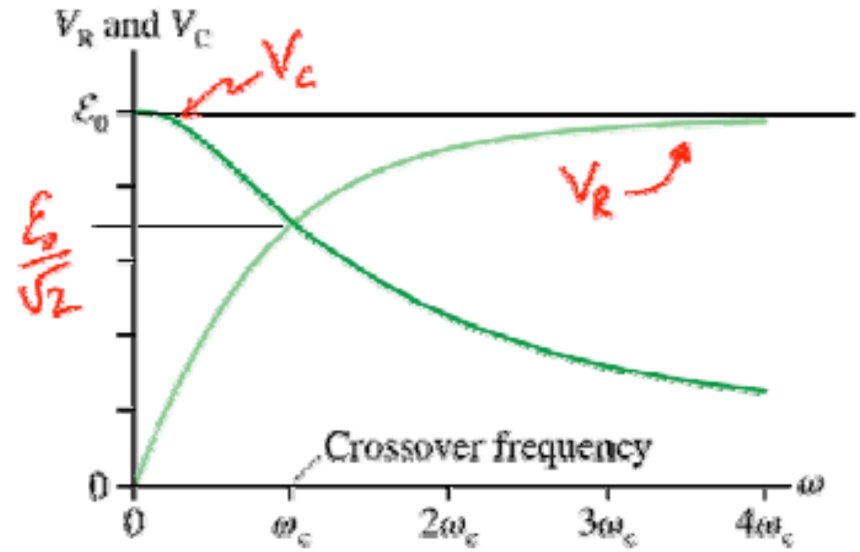


RC Filters – Analysis

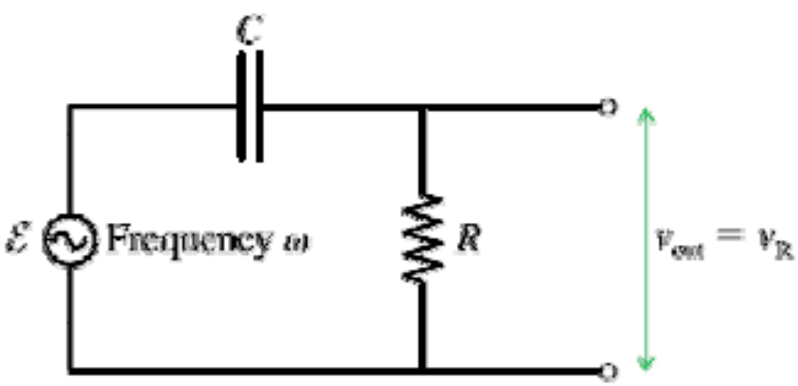
(a) Low-pass filter



Transmits frequencies $\omega < \omega_c$ and blocks frequencies $\omega > \omega_c$.



(b) High-pass filter



Transmits frequencies $\omega > \omega_c$ and blocks frequencies $\omega < \omega_c$.

- Capacitor like a short at high frequencies since:

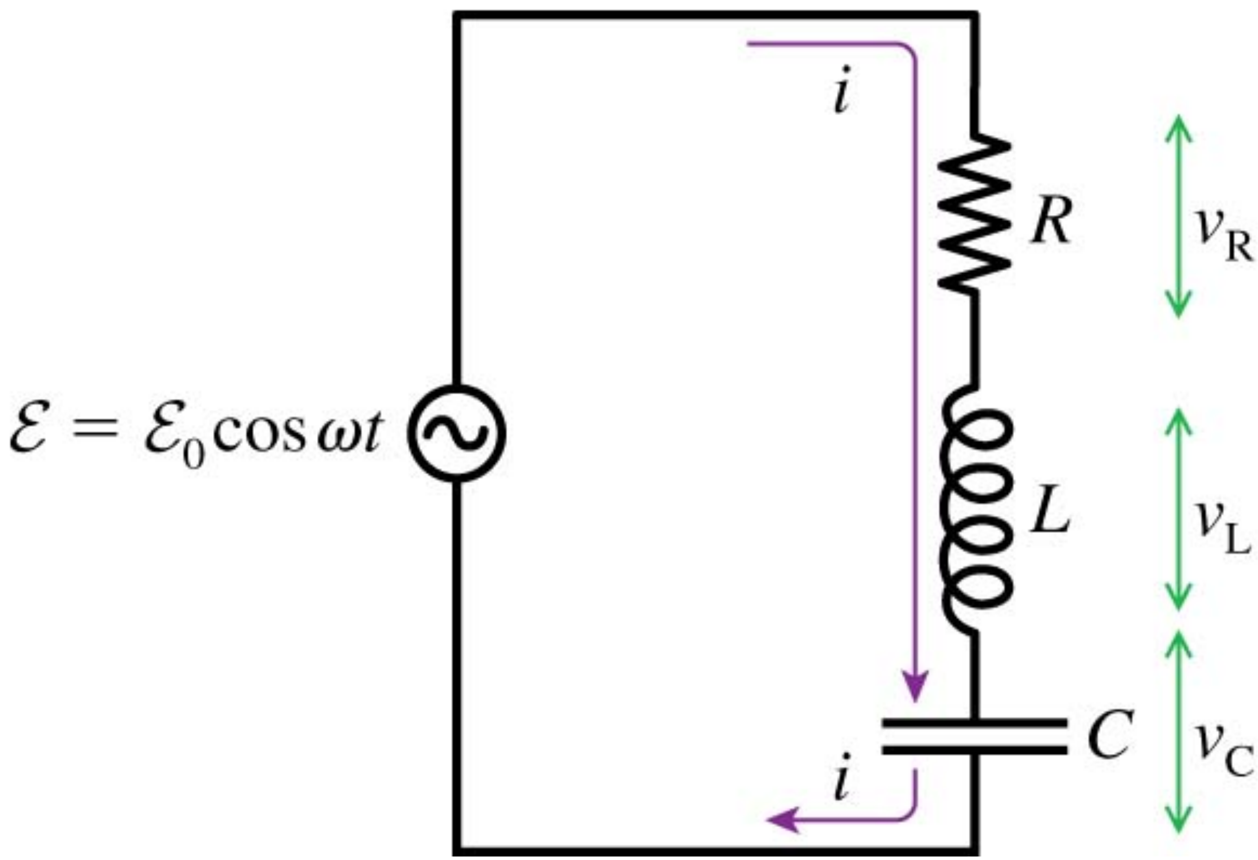
$$X_C = \frac{1}{wC} \longrightarrow 0 \text{ at High } w$$

- Voltage across Capacitor dominates at low frequencies since:

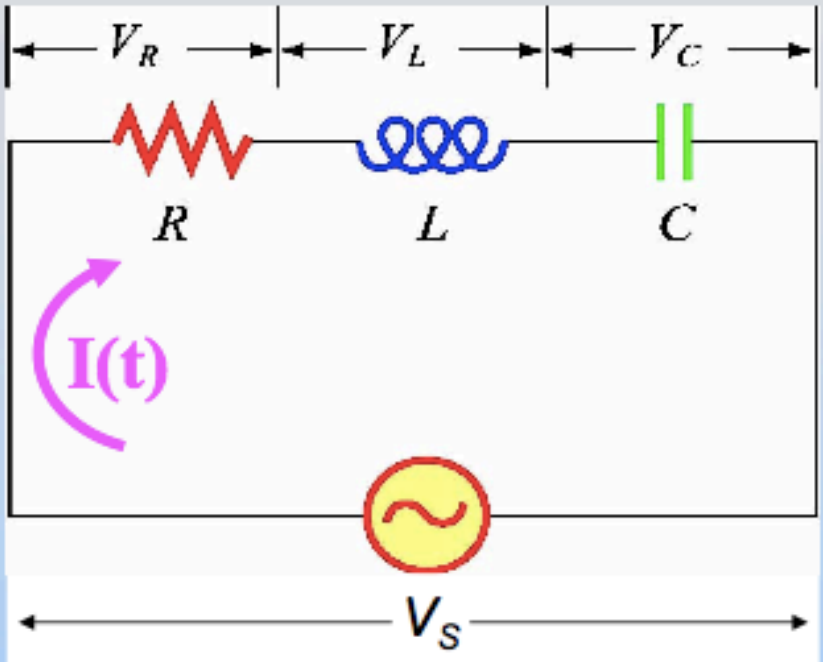
$$X_C = \frac{1}{wC} \Rightarrow X_C \gg R \text{ as } w \to 0$$

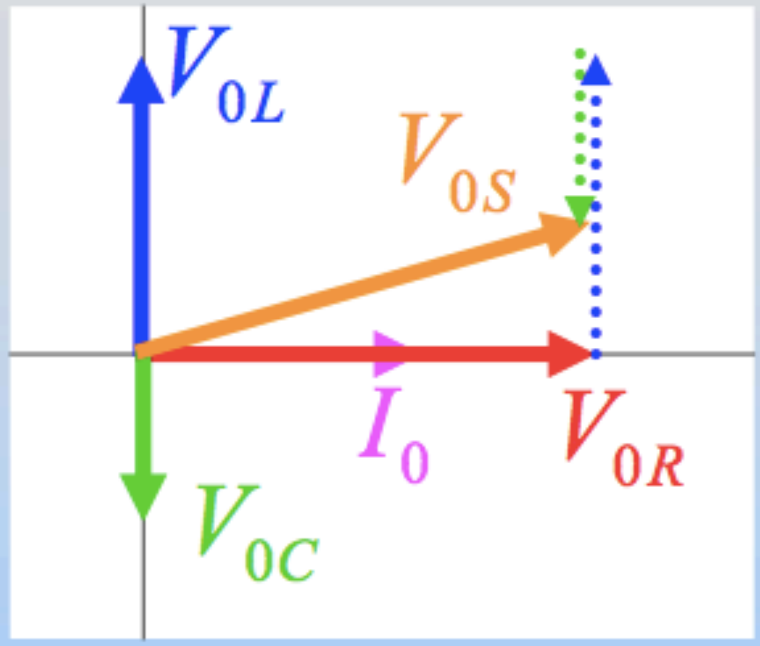
- If you input music, voltage across resistor would be like treble and voltage across capacitor would be like bass. Build your own speaker cross-over for woofer and tweeter.

FIGURE 36.17 A series *RLC* circuit.



Driven RLC Series Circuit

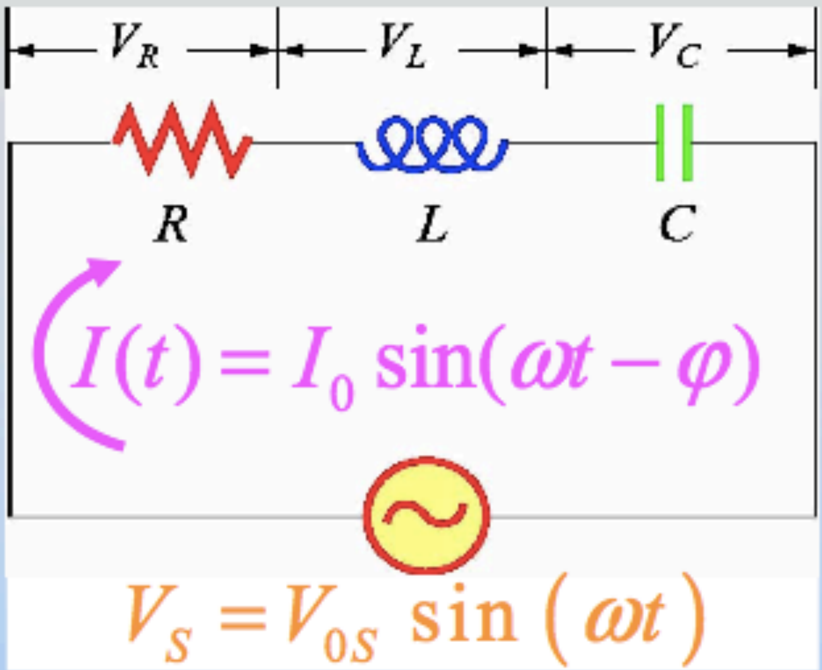


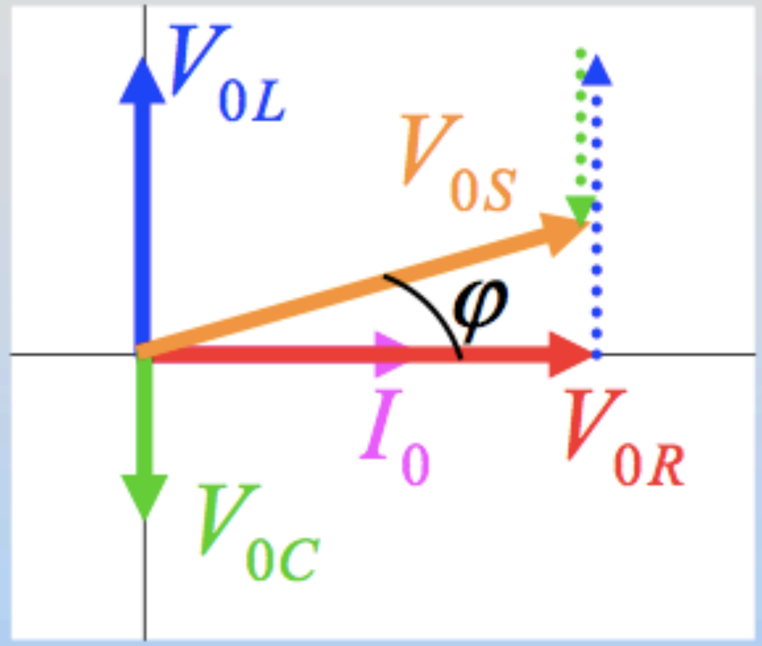


Now Solve: $V_S = V_R + V_L + V_C$

Now we just need to read the phasor diagram!

Driven RLC Series Circuit





$$V_{0S} = \sqrt{V_{R0}{}^2 + (V_{L0} - V_{C0})^2} = I_0\sqrt{R^2 + (X_L - X_C)^2} \equiv I_0 Z$$

$$I_0 = \frac{V_{0S}}{Z}$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Impedance

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Question of Phase

We had fixed phase of voltage:

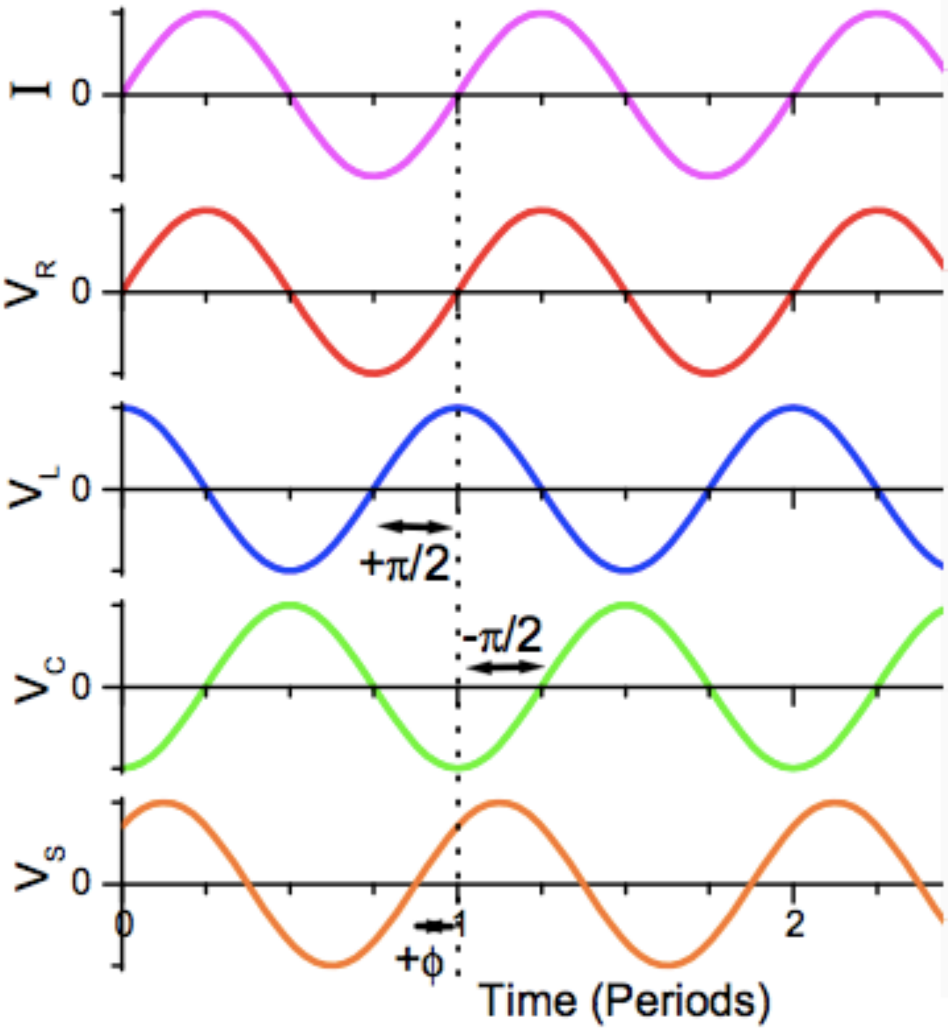
$$V = V_0 \sin \omega t \qquad I(t) = I_0 \sin(\omega t - \phi)$$

It's the same to write:

$$V = V_0 \sin(\omega t + \phi) \quad I(t) = I_0 \sin \omega t$$

(Just shifting zero of time)

Plot I, V's vs. Time



$$I(t) = I_0 \sin(\omega t)$$

$$V_R(t) = I_0 R \sin(\omega t)$$

$$V_L(t) = I_0 X_L \sin\left(\omega t + \tfrac{\pi}{2}\right)$$

$$V_L(t) = I_0 X_C \sin\left(\omega t - \tfrac{\pi}{2}\right)$$

$$V_S(t) = V_{S0} \sin(\omega t + \varphi)$$

Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$

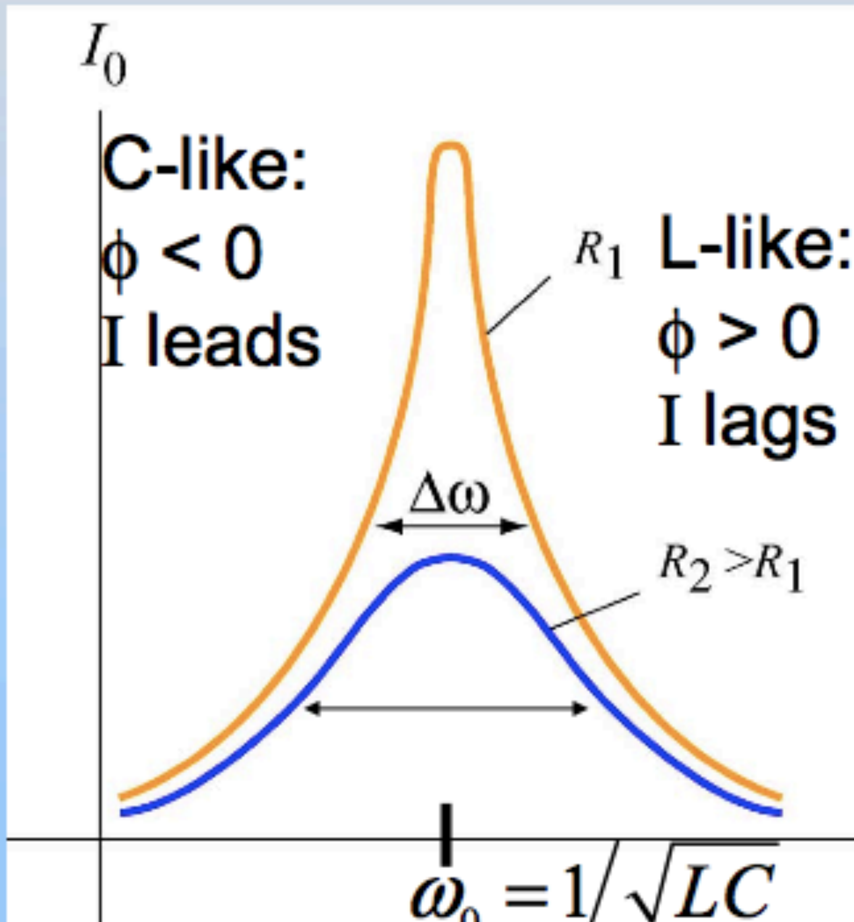
At very low frequencies, C dominates ($X_C >> X_L$):
it fills up and keeps the current low
At very high frequencies, L dominates ($X_L >> X_C$):
the current tries to change but it won't let it
At intermediate frequencies we have **resonance**

I_0 reaches maximum when $X_L = X_C$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

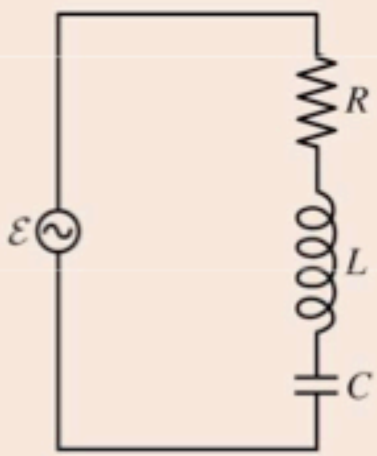
Resonance

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}}; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C}$$



$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Series *RLC* circuits

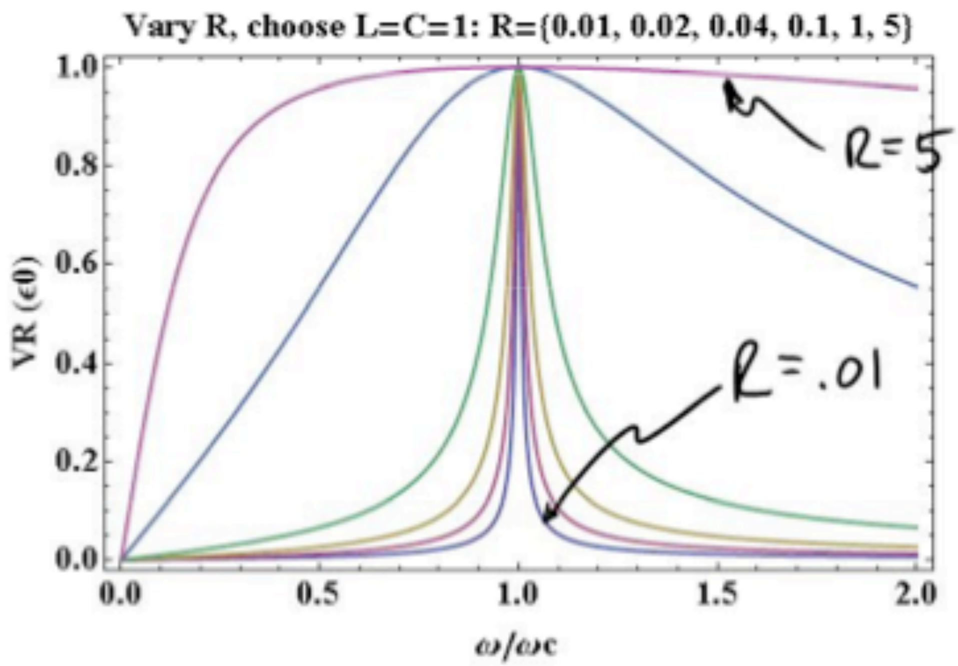


$I = \mathcal{E}_0/Z$ where Z is the **impedance**

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$V_R = IR \qquad V_L = IX_L \qquad V_C = IX_C$$

When $\omega = \omega_0 = 1/\sqrt{LC}$ (the **resonance frequency**), the current in the circuit is a maximum $I_{max} = \mathcal{E}_0/R$.



Problem Solving Tips

In this chapter, we have seen how phasors provide a powerful tool for analyzing the AC circuits. Below are some important tips:

1. Keep in mind the phase relationships for simple circuits

   (1) For a resistor, the voltage and the phase are always in phase.
   (2) For an inductor, the current lags the voltage by $90°$.
   (3) For a capacitor, the current leads to voltage by $90°$.

2. When circuit elements are connected in *series*, the instantaneous current is the same for all elements, and the instantaneous voltages across the elements are out of phase. On the other hand, when circuit elements are connected in *parallel*, the instantaneous voltage is the same for all elements, and the instantaneous currents across the elements are out of phase.

3. For series connection, draw a phasor diagram for the voltages. The amplitudes of the voltage drop across all the circuit elements involved should be represented with phasors. In Figure 12.8.1 the phasor diagram for a series *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

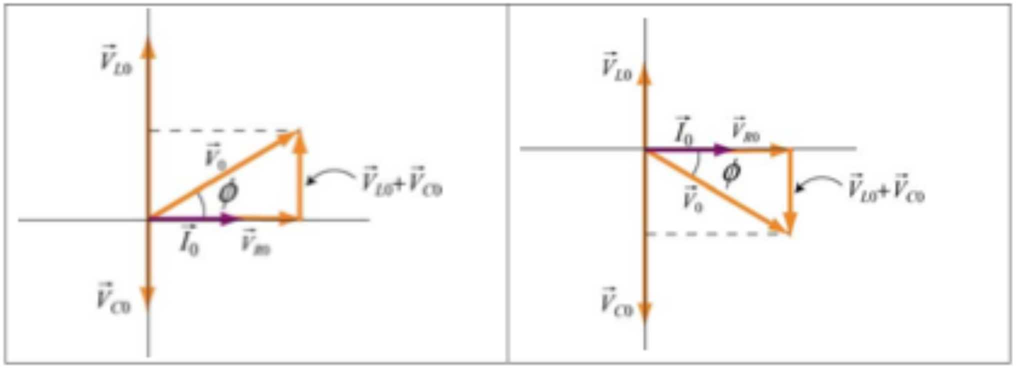


Figure 12.8.1 Phasor diagram for the series *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.1(a), we see that $V_{L0} > V_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. On the other hand, in the capacitive case shown in Figure 12.8.1(b), $V_{C0} > V_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ.

4. When $V_{L0} = V_{C0}$, or $\phi = 0$, the circuit is at resonance. The corresponding resonant frequency is $\omega_0 = 1/\sqrt{LC}$, and the power delivered to the resistor is a maximum.

5. For parallel connection, draw a phasor diagram for the currents. The amplitudes of the currents across all the circuit elements involved should be represented with phasors. In Figure 12.8.2 the phasor diagram for a parallel *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

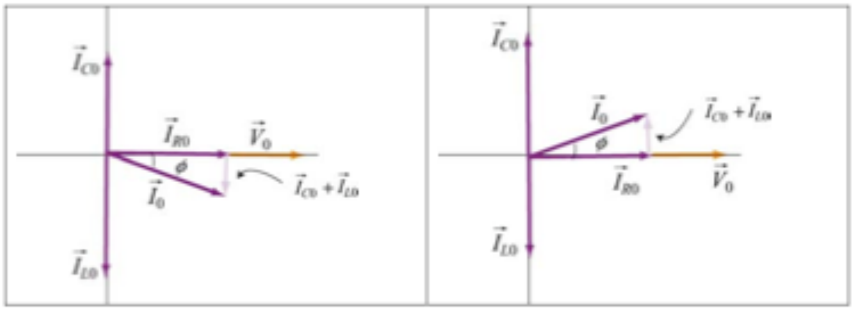


Figure 12.8.2 Phasor diagram for the parallel *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.2(a), we see that $I_{L0} > I_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. On the other hand, in the capacitive case shown in Figure 12.8.2(b), $I_{C0} > I_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ.

Example

Consider the circuit shown in Figure 12.9.3. The sinusoidal voltage source is $V(t) = V_0 \sin \omega t$. If both switches S_1 and S_2 are closed initially, find the following quantities, ignoring the transient effect and assuming that R, L, V_0 and ω are known:

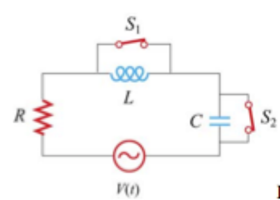


Figure 12.9.3

(a) the current $I(t)$ as a function of time,

(b) the average power delivered to the circuit,

(c) the current as a function of time a long time after only S_1 is opened.

(d) the capacitance C if both S_1 and S_2 are opened for a long time, with the current and voltage in phase.

(e) the impedance of the circuit when both S_1 and S_2 are opened.

(f) the maximum energy stored in the capacitor during oscillations.

(g) the maximum energy stored in the inductor during oscillations.

(h) the phase difference between the current and the voltage if the frequency of $V(t)$ is doubled.

(i) the frequency at which the inductive reactance X_L is equal to half the capacitive reactance X_C.

Solutions:

(a) When both switches S_1 and S_2 are closed, the current goes through only the generator and the resistor, so the total impedance of the circuit is R and the current is

$$I_R(t) = \frac{V_0}{R}\sin\omega t \qquad (12.9.25)$$

(b) The average power is given by

$$\langle P(t)\rangle = \langle I_R(t)V(t)\rangle = \frac{V_0^2}{R}\langle \sin^2\omega t\rangle = \frac{V_0^2}{2R} \qquad (12.9.26)$$

(c) If only S_1 is opened, after a long time the current will pass through the generator, the resistor and the inductor. For this RL circuit, the impedance becomes

$$Z = \frac{1}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \qquad (12.9.27)$$

and the phase angle ϕ is

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) \qquad (12.9.28)$$

Thus, the current as a function of time is

$$I(t) = I_0\sin(\omega t - \phi) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right) \qquad (12.9.29)$$

Note that in the limit of vanishing resistance $R = 0$, $\phi = \pi/2$, and we recover the expected result for a purely inductive circuit.

(d) If both switches are opened, then this would be a driven *RLC* circuit, with the phase angle ϕ given by

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - \dfrac{1}{\omega C}}{R} \tag{12.9.30}$$

If the current and the voltage are in phase, then $\phi = 0$, implying $\tan\phi = 0$. Let the corresponding angular frequency be ω_0; we then obtain

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{12.9.31}$$

and the capacitance is

$$C = \frac{1}{\omega_0^{\,2} L} \tag{12.9.32}$$

(e) From (d), we see that when both switches are opened, the circuit is at resonance with $X_L = X_C$. Thus, the impedance of the circuit becomes

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = R \tag{12.9.33}$$

(f) The electric energy stored in the capacitor is

$$U_E = \frac{1}{2} C V_C^2 = \frac{1}{2} C (I X_C)^2 \tag{12.9.34}$$

It attains maximum when the current is at its maximum I_0:

$$U_{C,\max} = \frac{1}{2} C I_0^2 X_C^2 = \frac{1}{2} C \left(\frac{V_0}{R}\right)^2 \frac{1}{\omega_0^{\,2} C^2} = \frac{V_0^2 L}{2R^2} \tag{12.9.35}$$

where we have used $\omega_0^2 = 1/LC$.

(g) The maximum energy stored in the inductor is given by

$$U_{L,\max} = \frac{1}{2} L I_0^2 = \frac{L V_0^2}{2R^2} \tag{12.9.36}$$

(h) If the frequency of the voltage source is doubled, i.e., $\omega = 2\omega_0 = 1/\sqrt{LC}$, then the phase becomes

$$\phi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) = \tan^{-1}\left(\frac{\left(2/\sqrt{LC}\right)L - \left(\sqrt{LC}/2C\right)}{R}\right) = \tan^{-1}\left(\frac{3}{2R}\sqrt{\frac{L}{C}}\right) \tag{12.9.37}$$

(i) If the inductive reactance is one-half the capacitive reactance,

$$X_L = \frac{1}{2} X_C \quad \Rightarrow \quad \omega L = \frac{1}{2}\left(\frac{1}{\omega C}\right) \tag{12.9.38}$$

then

$$\omega = \frac{1}{\sqrt{2LC}} = \frac{\omega_0}{\sqrt{2}} \tag{12.9.39}$$