PHYSICS 270
Review LR and LC Circuits
Feb. 17 2011

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Refer to: http://ocw.mit.edu/courses/physics/
8-02-physics-ii-electricity-and-magnetism-
spring-2007/class-slides/

- The **self-inductance** of a coil with N turns is

$$L = \frac{N\Phi_B}{I}$$

where Φ_B is the magnetic flux through one turn of the coil.

- The **self-induced emf** responding to a change in current inside a coil current is

$$\varepsilon_L = -L\frac{dI}{dt}$$

- The inductance of a solenoid with N turns, cross sectional area A and length l is

$$L = \frac{\mu_0 N^2 A}{l}$$

- If a battery supplying an emf ε is connected to an inductor and a resistor in series at time $t = 0$, then the current in this ***RL* circuit** as a function of time is

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$$

where $\tau = L/R$ is the time constant of the circuit. If the battery is removed in the *RL* circuit, the current will decay as

$$I(t) = \left(\frac{\varepsilon}{R}\right)e^{-t/\tau}$$

- The **magnetic energy** stored in an inductor with current I passing through is

$$U_B = \frac{1}{2}LI^2$$

- The **magnetic energy density** at a point with magnetic field B is

$$u_B = \frac{B^2}{2\mu_0}$$

- The differential equation for an oscillating ***LC* circuit** is

$$\frac{d^2Q}{dt^2} + \omega_0^2 Q = 0$$

where $\omega_0 = \dfrac{1}{\sqrt{LC}}$ is the angular frequency of oscillation. The charge on t capacitor as a function of time is given by

$$Q(t) = Q_0 \cos(\omega_0 t + \phi)$$

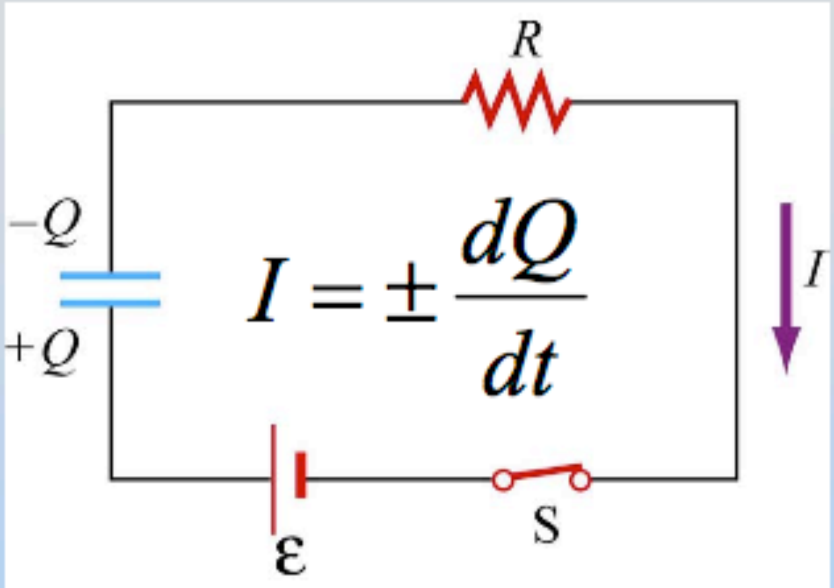
and the current in the circuit is

$$I(t) = -\frac{dQ}{dt} = +\omega_0 Q_0 \sin(\omega_0 t + \phi)$$

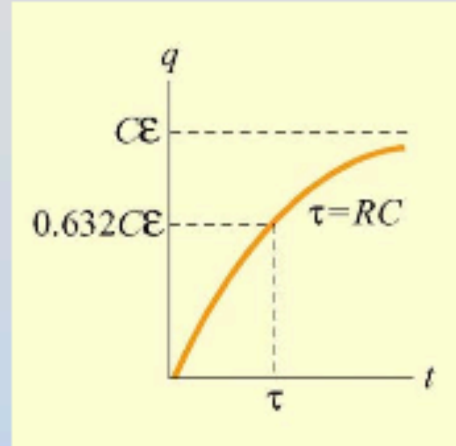
- The total energy in an LC circuit is, using $I_0 = \omega_0 Q_0$,

$$U = U_E + U_B = \frac{Q_0^2}{2C}\cos^2\omega_0 t + \frac{LI_0^2}{2}\sin^2\omega_0 t = \frac{Q_0^2}{2C}$$

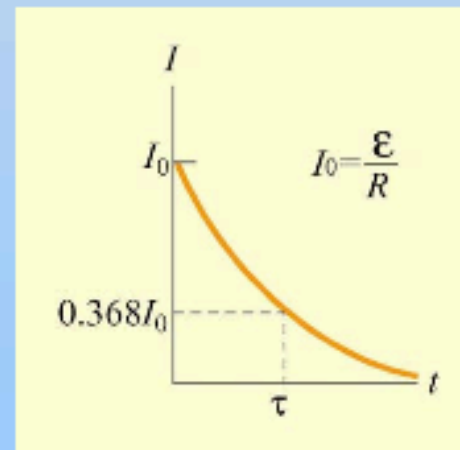
(Dis)Charging A Capacitor



$$I = \pm \frac{dQ}{dt}$$



$$Q = C\mathcal{E}\left(1 - e^{-t/RC}\right)$$



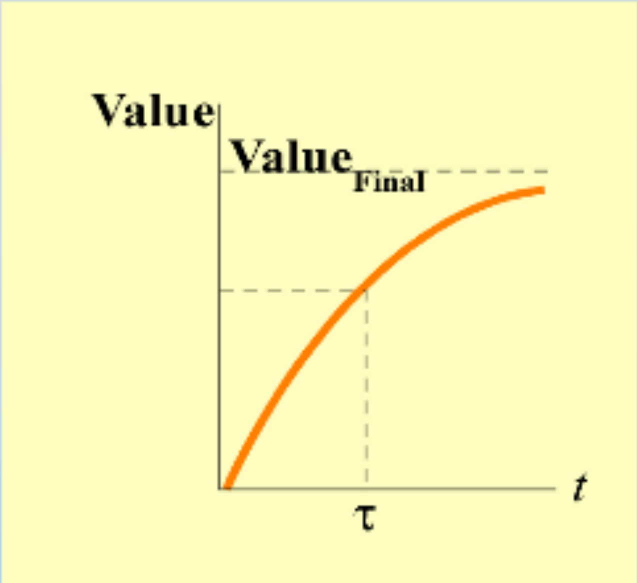
$$I = \frac{dQ}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC}$$

$$\sum_i \Delta V_i = \varepsilon - \frac{Q}{C} - IR = 0$$

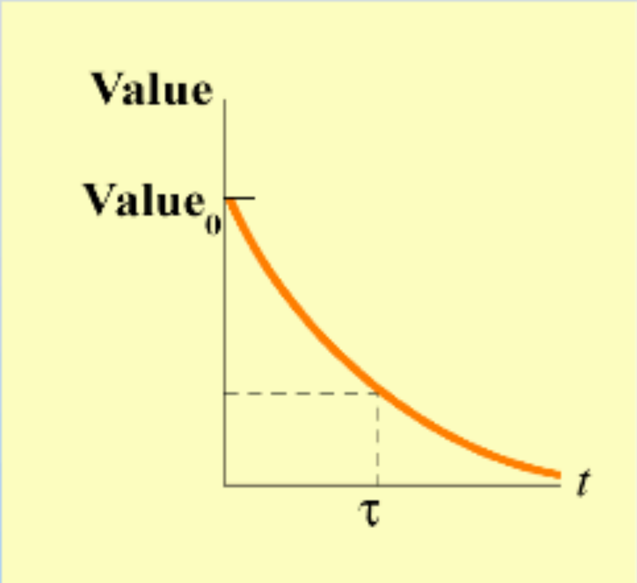
$$\overbrace{\boxed{C\mathcal{E}}}^{Q_{final}} - Q - \overbrace{\boxed{RC}}^{\tau} \frac{dQ}{dt} = 0$$

General Comment: RC

All Quantities Either:



$$\text{Value}(t) = \text{Value}_{\text{Final}}\left(1 - e^{-t/\tau}\right)$$



$$\text{Value}(t) = \text{Value}_0 e^{-t/\tau}$$

τ can be obtained from differential equation (prefactor on d/dt) e.g. τ = RC

Kirchhoff's Loop Rule Modified for Inductors:

If an inductor is traversed in the direction of the current, the "potential change" is $-L(dI/dt)$. On the other hand, if the inductor is traversed in the direction opposite of the current, the "potential change" is $+L(dI/dt)$.

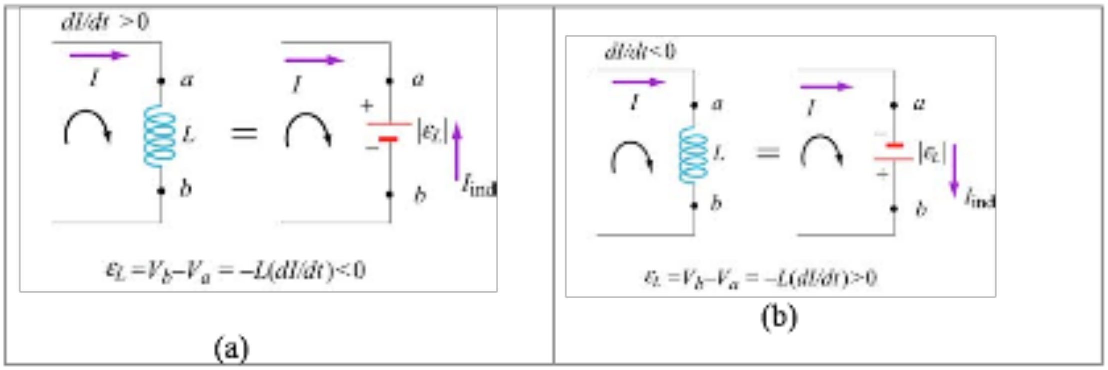


Figure 11.4.2 Modified Kirchhoff's rule for inductors (a) with increasing current, and (b) with decreasing current.

The modified rule for inductors may be obtained as follows: The polarity of the self-induced emf is such as to oppose the change in current, in accord with Lenz's law. If the rate of change of current is positive, as shown in Figure 11.4.2(a), the self-induced emf ε_L sets up an induced current I_{ind} moving in the opposite direction of the current I to oppose such an increase. The inductor could be replaced by an emf $|\varepsilon_L| = L|dI/dt| = +L(dI/dt)$ with the polarity shown in Figure 11.4.2(a). On the other hand, if $dI/dt < 0$, as shown in Figure 11.4.2(b), the induced current I_{ind} set up by the self-induced emf ε_L flows in the same direction as I to oppose such a decrease.

We see that whether the rate of change of current in increasing ($dI/dt > 0$) or decreasing ($dI/dt < 0$), in both cases, the change in potential when moving from a to b along the direction of the current I is $V_b - V_a = -L(dI/dt)$. Thus, we have

11.4.2 Rising Current

Consider the *RL* circuit shown in Figure 11.4.3. At $t = 0$ the switch is closed. We find that the current does not rise immediately to its maximum value ε / R. This is due to the presence of the self-induced emf in the inductor.

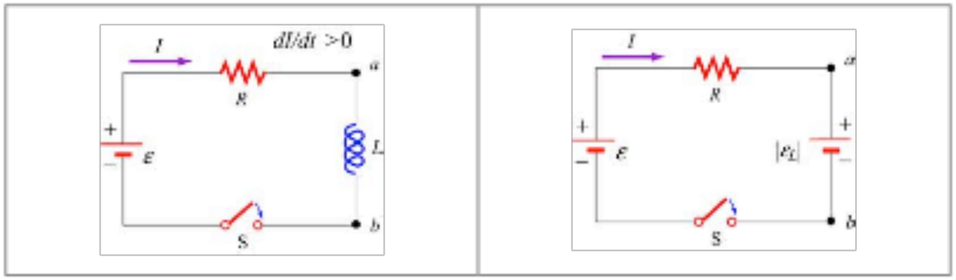


Figure 11.4.3 (a) *RL* Circuit with rising current. (b) Equivalent circuit using the modified Kirchhoff's loop rule.

Using the modified Kirchhoff's rule for increasing current, $dI/dt > 0$, the *RL* circuit is described by the following differential equation:

$$\varepsilon - IR - |\varepsilon_L| = \varepsilon - IR - L\frac{dI}{dt} = 0 \qquad (11.4.5)$$

Note that there is an important distinction between an inductor and a resistor. The potential difference across a resistor depends on *I*, while the potential difference across an inductor depends on dI/dt. The self-induced emf does not oppose the current itself, but the change of current dI/dt.

The above equation can be rewritten as

$$\frac{dI}{I - \varepsilon / R} = -\frac{dt}{L/R} \qquad (11.4.6)$$

Integrating over both sides and imposing the condition $I(t=0) = 0$, the solution to the differential equation is

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right) \qquad (11.4.7)$$

where

$$\tau = \frac{L}{R} \tag{11.4.8}$$

is the time constant of the *RL* circuit. The qualitative behavior of the current as a function of time is depicted in Figure 11.4.4.

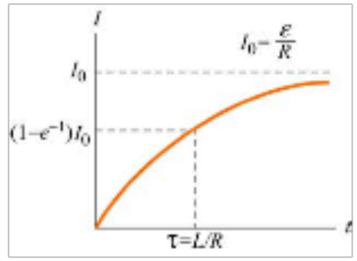


Figure 11.4.4 Current in the *RL* circuit as a function of time

Note that after a sufficiently long time, the current reaches its equilibrium value ε / R. The time constant τ is a measure of how fast the equilibrium state is attained; the larger the value of *L*, the longer it takes to build up the current. A comparison of the behavior of current in a circuit with or without an inductor is shown in Figure 11.4.5 below.

Similarly, the magnitude of the self-induced emf can be obtained as

$$|\varepsilon_L| = \left| -L\frac{dI}{dt} \right| = \varepsilon e^{-t/\tau} \tag{11.4.9}$$

which is at a maximum when $t = 0$ and vanishes as t approaches infinity. This implies that a sufficiently long time after the switch is closed, self-induction disappears and the inductor simply acts as a conducting wire connecting two parts of the circuit.

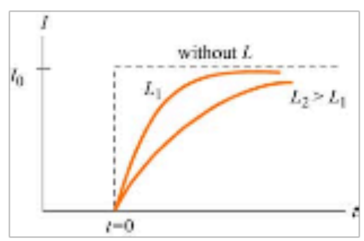


Figure 11.4.5 Behavior of current in a circuit with or without an inductor

11.4.3 Decaying Current

Next we consider the *RL* circuit shown in Figure 11.4.6. Suppose the switch S_1 has been closed for a long time so that the current is at its equilibrium value ε / R. What happens to the current when at $t = 0$ switches S_1 is opened and S_2 closed?

Applying modified Kirchhoff's loop rule to the right loop for decreasing current, $dI/dt < 0$, yields

$$|\varepsilon_L| - IR = -L\frac{dI}{dt} - IR = 0 \qquad (11.4.11)$$

which can be rewritten as

$$\frac{dI}{I} = -\frac{dt}{L/R} \qquad (11.4.12)$$

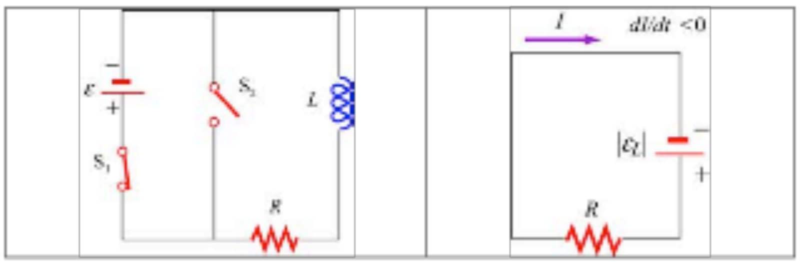


Figure 11.4.6 (a) *RL* circuit with decaying current, and (b) equivalent circuit.

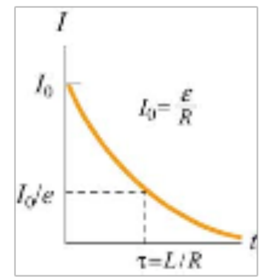


Figure 11.4.7 Decaying current in an *RL* circuit

The solution to the above differential equation is

$$I(t) = \frac{\varepsilon}{R} e^{-t/\tau} \qquad (11.4.13)$$

where $\tau = L/R$ is the same time constant as in the case of rising current. A plot of the current as a function of time is shown in Figure 11.4.7.

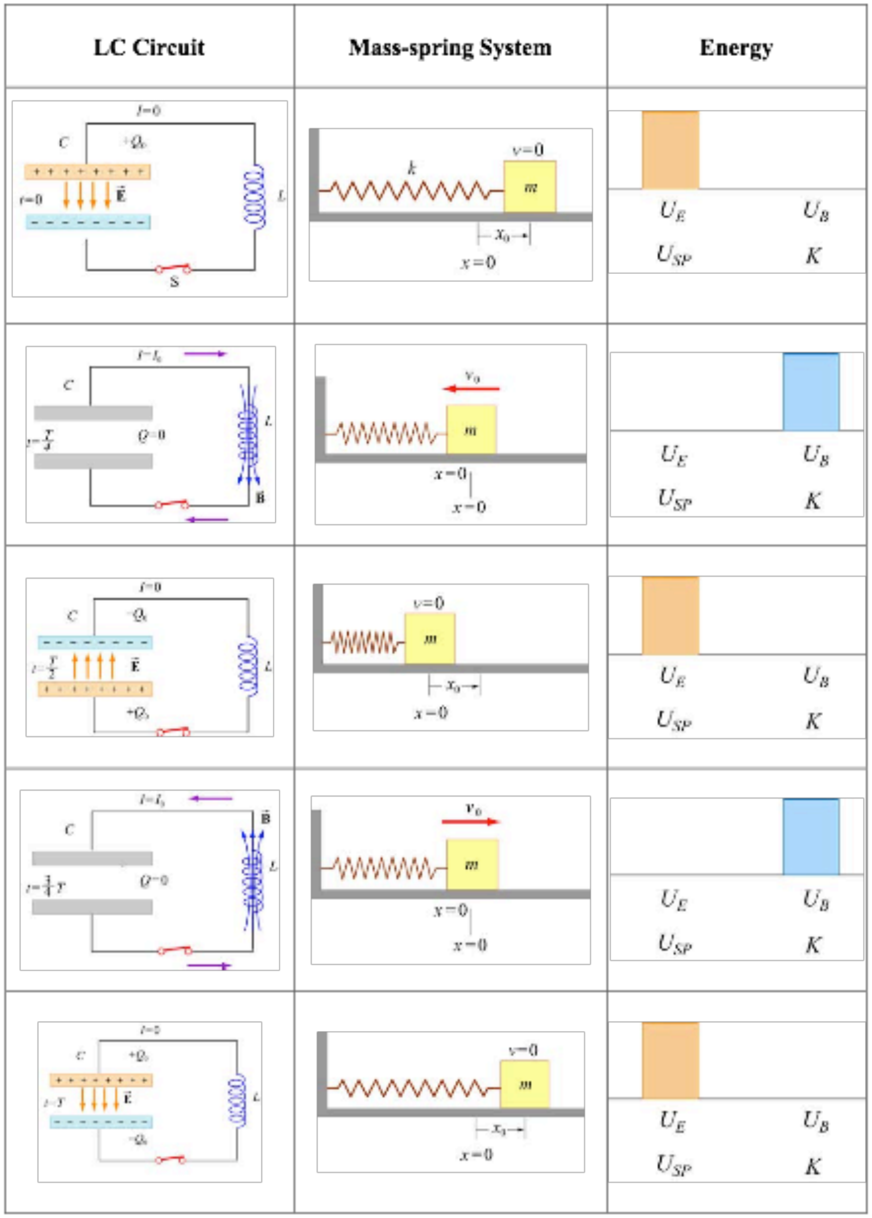


Figure 11.5.5 Energy oscillations in the *LC* Circuit and the mass-spring system

- The differential equation for an oscillating ***LC* circuit** is

$$\frac{d^2Q}{dt^2}+\omega_0^2 Q = 0$$

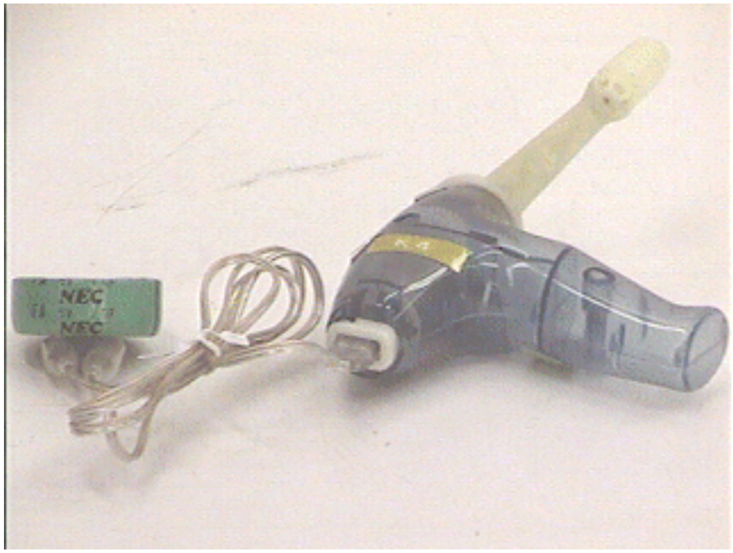
where $\omega_0=\dfrac{1}{\sqrt{LC}}$ is the angular frequency of oscillation. The charge on the capacitor as a function of time is given by

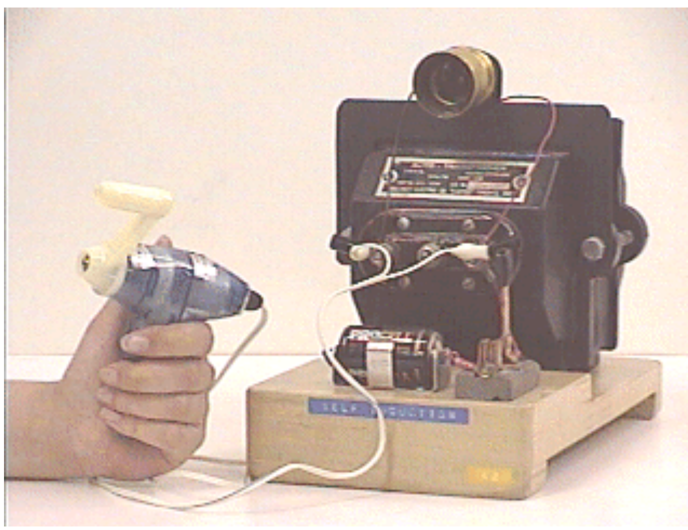
$$Q(t)=Q_0\cos(\omega_0 t+\phi)$$

and the current in the circuit is

$$I(t)=-\frac{dQ}{dt}=+\omega_0 Q_0\sin(\omega_0 t+\phi)$$

In same direction





Opposite direction

Consider a wire aligned perpendicular to a constant magnetic field moving along a path perpendicular to the magnetic field lines: a current will be generated in the wire flowing in the direction determined by the **v x B** force on the free electrons in the wire. A current in the opposite direction through the wire would experience a force in the direction the original wire was moving.
The capacitor is charged as the Genecon generates current due to a coil moving in a magnetic field. When the cranking stops, the capacitor discharges by a current in the opposite direction to the current that charged it. This current interacts with the magnetic field of the genecon to create a force on the coils in the same direction as the original cranking motion.

When the handle is rapidly cranked, it creates a potential difference across the inductor and therefore creates an electrical current through its windings. This creates a magnetic field in the core that stores electrical energy in the form of a magnetic field. When the magnetic field collapses, it creates an electric field in the same direction as the original electric field from the generator that caused the current through the windings. This makes the generator, now operating as a motor, move in the opposite direction.