OVERVIEW

- COMPARE E-FIELD TO B-FIELD PROPERTIES
  - FIELDS, FIELD LINES, DIPOLES

- COMPUTE B-FIELDS DUE TO CURRENTS
  - BIOT-SAVART, LONG WIRES, CURRENT LOOPS, SOLENOIDS
  - AMPERE’s LAW – EQUIVALENCE TO GAUSS’s LAW

- MAGNETIC FORCES ON MOVING CHARGES

- MAGNETIC PROPERTIES OF MATTER - MRI
LAST TIME

BIOT-SAVART LAW

\[ \vec{B}_{\text{point charge}} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2} \]  
(magnetic field of a point charge)

Direction by RHS rule
Field due to a long wire

\[ \Delta \vec{B}_{\Delta z}(\vec{r}) = k_M \frac{I \Delta z (\hat{e}_z \times \vec{r})}{r^3} \]

\[ \vec{r} = \rho \hat{e}_\rho + z \hat{e}_z \]

\[ r^2 = \rho^2 + z^2 \]

\[ \Delta \vec{B}_{\Delta z}(\vec{r}) = k_M \frac{I \Delta z (\hat{e}_z \times \vec{r})}{r^3} = k_M \frac{I \Delta z (\hat{e}_z \times (\hat{e}_\rho \rho + \hat{e}_z z))}{r^3} = k_M \frac{I \Delta z \rho (\hat{e}_z \times \hat{e}_\rho)}{r^3} = k_M \frac{I \Delta z \rho \hat{e}_\theta}{r^3} \]

\[ \vec{B}(\vec{r}) = \hat{e}_\theta (k_M I \rho) \int_{-l}^{l} \frac{dz}{(\rho^2 + z^2)^{3/2}} = \hat{e}_\theta (k_M I \rho) \left[ \frac{z}{\rho^2(\rho^2 + z^2)^{1/2}} \right]^{l}_{-l} \]

\[ l \to \infty, \rho \ll l \]

\[ \vec{B}(\vec{r}) = \hat{e}_\theta k_M (2I / \rho) \]
Biot-Savart: Circular Arc $\rightarrow$ Circle

Current element of length $d\mathbf{s}$ carrying current $I$ produces a magnetic field:

\[
\mathbf{dB} = \frac{\mu_0}{4\pi} \frac{I}{r^2} d\mathbf{s} \times \hat{\mathbf{r}}
\]

For 1 and 3

$d\mathbf{s}$ parallel to $\hat{\mathbf{r}} \rightarrow d\mathbf{s} \times \hat{\mathbf{r}} = 0$

For 2 $\rightarrow d\mathbf{s} \times \hat{\mathbf{r}} = ds$ direction into the page

\[ds = Rd\theta\]

\[|\mathbf{B}| = \frac{\mu_0}{4\pi} I \int_{0}^{\theta} \frac{Rd\theta}{R^2} = \frac{\mu_0}{4\pi} \frac{I}{R} \theta\]

Circle: $\theta \rightarrow 2\pi$

\[|\mathbf{B}| = \frac{\mu_0}{2} I\]
Consider a coil with radius $R$ and current $I$

Find the magnetic field $B$ at the center ($P$)

1) Think about it:
   - Legs contribute nothing
     $I$ parallel to $r$
   - Ring makes field into page
2) Choose a $ds$
3) Pick your coordinates
4) Write Biot-Savart

Current element of length $ds$ carrying current $I$ produces a magnetic field:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2}$$

In the circular part of the coil...

$$d\vec{s} \perp \hat{r} \quad \Rightarrow \quad I \, d\vec{s} \times \hat{r} = ds$$

Biot-Savart:

$$dB = \frac{\mu_0 I \, |d\vec{s} \times \hat{r}|}{4\pi r^2} = \frac{\mu_0 I \, ds}{4\pi r^2}$$

$$= \frac{\mu_0 I}{4\pi} \frac{R \, d\theta}{R^2}$$

$$= \frac{\mu_0 I}{4\pi} \frac{d\theta}{R}$$
**B field of a current loop**

In the circular part of the coil...

\[ d\hat{s} \perp \hat{r} \rightarrow \| d\hat{s} \times \hat{r} \| = ds \]

**Biot-Savart:**

\[
\begin{align*}
    dB &= \frac{\mu_0 I |d\hat{s} \times \hat{r}|}{4\pi r^2} = \frac{\mu_0 I}{4\pi r^2} ds \\
    &= \frac{\mu_0 I R \, d\theta}{4\pi R^2} \\
    &= \frac{\mu_0 I}{4\pi} \frac{d\theta}{R}
\end{align*}
\]

Consider a coil with radius \( R \) and current \( I \)

\[
\begin{align*}
    dB &= \frac{\mu_0 I}{4\pi} \frac{d\theta}{R} \\
    B &= \int dB = \int_0^{2\pi} \frac{\mu_0 I}{4\pi} \frac{d\theta}{R} \\
    &= \frac{\mu_0 I}{4\pi R} \int_0^{2\pi} d\theta = \frac{\mu_0 I}{4\pi R} (2\pi) \\
    \bar{B} &= \frac{\mu_0 I}{2R} \text{ into page}
\end{align*}
\]
Applying the Biot-Savart Law:
On-axis of circular loop

\[ dB = \frac{M_0 I d\hat{S} \times \hat{r}}{4\pi r^2} \]

When we integrate, \( B_y \to 0 \) and all \( x \)-components add.

Let \( \theta \) be the angle between \( dB \) and \( \hat{x} \):

\[ |dB| = dB \cdot \hat{x} = |dB| \cos \theta \]

\[ \cos \theta = \frac{R}{r}, \quad r = \sqrt{x^2 + R^2}, \quad d\hat{S} = \hat{r} \]

\[ => |d\hat{S}| = R \, d\phi \]

\[ |B| = \int |dB| = \frac{M_0 I}{4\pi} \int \frac{R \, d\phi}{r^2} \frac{R}{r} \]

\[ B = \frac{M_0 I}{2} \frac{R^2}{(x^2 + R^2)^{3/2}} \hat{x} \]
Applying the Biot-Savart Law:
On-axis of circular loop

\[ \vec{B} = \frac{M_o}{2} I \frac{R^2}{(x^2 + R^2)^{3/2}} \hat{x} \]

\[ \lim_{x \to 0} \vec{B} = \frac{M_o}{2} \frac{I}{R} \hat{x} \]

Let \( x \gg R \), + (\( x^2 + R^2 \))^{3/2} = \( x^3 \left(1 + \frac{R^2}{x^2} \right)^{3/2} \)

\[ \Rightarrow \vec{B} = \frac{M_o}{4 \pi} 2 \frac{I \pi R^2 \hat{z}}{x^3} \]

\[ \Rightarrow \vec{B} = \frac{M_o}{4 \pi} \frac{2 \hat{z}}{x^3} \text{ on axis} \]
Applying the Biot-Savart Law:
On-axis of circular loop

\[ \vec{B} = \frac{\mu_0}{4\pi} \frac{2\vec{M}}{x^3} \text{ on axis} \]

Looks like Electric dipole: \( \vec{p} = q\vec{s} \)

\[ \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\vec{p}}{x^3} \]

Remember from Electrostatics

\[ U_{E} = -\vec{p} \cdot \vec{E} \]
\[ U_{M} = -\vec{\mu} \cdot \vec{B} \]
\[ \tau_{E} = \vec{p} \times \vec{E} \]
\[ \tau_{B} = \vec{\mu} \times \vec{B} \]
Compute on axis B for $z \gg R$

$\mathbf{B}_k$ perpendicular to $\Delta s_k$ and $r$. The $\rho$-component of of $\mathbf{B}_k$ is cancelled by the $\rho$ component of $\mathbf{B}_j$. Same for all anti-diametric segments. We are thus left by only a $z$-component. Need to compute only $(B_k)_z = B_k \cos \phi$.

**Magnetic Dipole moment**

$$B_{loop} = \frac{\mu_0}{4\pi} \frac{2I\pi R^2}{z^3} = \frac{\mu_0}{4\pi} \frac{2IA}{z^3} = \frac{\mu_0}{4\pi} \frac{2I\mu}{z^3}$$

$$\vec{\mu} = \vec{A}I$$

$$\vec{E}_{dipole} = \frac{1}{4\pi \epsilon_0} \frac{2\vec{p}}{z^3}$$
MAGNETIC DIPOLE

**Figure 35.18** The magnetic field of a current loop.

(a) Cross section through the current loop   (b) The current loop seen from the right

The field emerges from the center of the loop.

The field returns around the outside of the loop.

(c) A photo of iron filings

The magnetic dipole moment is perpendicular to the loop, in the direction of the right-hand rule. The magnitude of \( \vec{\mu} \) is \( AI \).

**Figure 35.19** A current loop has magnetic poles and generates the same magnetic field as a flat permanent magnet.

(a) Current loop   (b) Permanent magnet

Whether it's a current loop or a permanent magnet, the magnetic field emerges from the north pole.

**Figure 37.8** The dipole moment.

\[ \vec{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0} \frac{2\vec{p}}{z^3}, \quad \vec{p} = e_z p, z >> l \]

\[ \vec{B}(0,0,z) = \frac{\mu_0}{4\pi} \frac{2\vec{\mu}}{z^3}, z >> R \]

\[ \mu = I\pi R^2 = I(Area) \]

**Note** The magnetic field inside a permanent magnet differs from the magnetic field at the center of a current loop. Only the exterior field of a magnet matches the field of a current loop.
A FLAT PERMANENT PERMANENT MAGNET AND A CURRENT LOOP (ELECTROMAGNET) GENERATE THE SAME MAGNETIC FIELD
THE FIELD OF A DIPOLE
EARTH’s FIELD – LOOP OR NOT?

The magnetic dipole moment is perpendicular to the loop, in the direction of the right-hand rule. The magnitude of \( \vec{\mu} \) is \( AI \).

\[
\vec{\mu}_{\text{earth}} = e \times 6.4 \times 10^{21} A - m^2
\]
Ampere’s Law
Gauss’s Law – The Idea

\[ \text{Flux} = \int \vec{E} \cdot d\vec{S} \]

The total “flux” of field lines penetrating any of these surfaces is the same and depends only on the amount of charge inside.
GAUSS’s LAW

\[ \Phi_c = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0} \]

**Figure 28.23** A spherical Gaussian surface surrounding a sphere of charge.

**Figure 28.24** A spherical Gaussian surface inside a uniform sphere of charge.

**Figure 28.25** The electric field strength of a uniform sphere of charge of radius \( R \).
Ampere’s Law: The Idea

In order to have a B field around a loop, there must be current punching through the loop.

Closed path made up of segments of length $\Delta s$.

Area enclosed by the path.
LINE INTEGRALS – AMPERE’s LAW

\[ l = \sum_{k} \Delta s_k \rightarrow \int_{i}^{f} ds \]

\[ \sum_{k} \vec{B}_k \cdot \Delta \vec{s}_k \rightarrow \int_{i}^{f} \vec{B} \cdot d\vec{s} = \text{the line integral of} \ \vec{B} \text{ from} \ i \ \text{to} \ f \]

**TACTICS**

**BOX 33.5** Evaluating line integrals

1. If \( \vec{B} \) is everywhere perpendicular to a line, the line integral of \( \vec{B} \) is
   \[ \int_{i}^{f} \vec{B} \cdot d\vec{s} = 0 \]

2. If \( \vec{B} \) is everywhere tangent to a line of length \( l \) and has the same magnitude \( B \) at every point, then
   \[ \int_{i}^{f} \vec{B} \cdot d\vec{s} = Bl \]

Exercises 23–24
Ampere’s Law: The Equation

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

The line integral is around any closed contour bounding an open surface $S$.

$I_{enc}$ is current through $S$:

$$I_{enc} = \int_S \vec{J} \cdot d\vec{A}$$
Calculate $A$ for each of the three loops

\[ A = \oint \vec{B}(\rho) \cdot d\vec{s} \]

\[ \vec{B}(\rho) = \hat{\rho} \frac{\mu_o I}{2\pi \rho} \]

Consider the following case for an infinite straight current carrying wire in $+\frac{x}{z}$ direction:

Loop 1
\[ A_1 = \oint_c \frac{\mu_o I}{2\pi \rho} \cdot (\hat{\rho} \rho d\theta + \hat{\rho} d\rho) = \frac{\mu_o I}{2\pi} (2\pi + 0) = \mu_o I \]

Loop 2
\[ A_2 = \oint_c \frac{\mu_o I}{2\pi \rho} \cdot (\hat{\rho} \rho d\theta + \hat{\rho} d\rho) = \frac{\mu_o I}{2\pi} [\oint d\theta + 0] = \mu_o I \]

Loop 3
\[ A_3 = \oint_{\theta_i}^{\theta_f} \frac{\mu_o I}{2\pi \rho} \cdot (\hat{\rho} \rho d\theta + \hat{\rho} d\rho) = \oint_{\theta_i}^{\theta_f} \frac{\mu_o I}{2\pi} d\theta \]
But $\theta_i = \theta_f \rightarrow A_3 = 0$
**Biot-Savart vs. Ampere**

| Biot-Savart Law | \[ \vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2} \] | general current source  
ex: finite wire wire loop |
|----------------|---------------------------------|---------------------|
| Ampere’s law   | \[ \int \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \] | symmetric current source  
ex: infinite wire infinite current sheet |
Applying Ampere’s Law

1. Identify regions in which to calculate B field
   Get B direction by right hand rule
2. Choose Amperian Loops S: Symmetry
   B is 0 or constant on the loop!
3. Calculate $\int \vec{B} \cdot d\vec{s}$
4. Calculate current enclosed by loop S
5. Apply Ampere’s Law to solve for B

$$\int \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$
Always True, Occasionally Useful

Like Gauss’s Law,

Ampere’s Law is **always** true

However, it is only useful for calculation in certain specific situations, involving highly symmetric currents.

Here are examples...
**AMPERE’s LAW**

**FIGURE 33.23** Integrating the magnetic field around a wire.

- The integration path is a circle of radius $d$.
- $\vec{B}$ is everywhere tangent to the integration path and has constant magnitude.

**FIGURE 33.24** Using Ampère’s law.

- $I_1$ doesn’t pass through the enclosed area.
- The integration path is a closed curve.
- These currents pass through the bounded area.

\[
(2\pi d)B_\theta = \mu_0 I \\
B_\theta = \frac{\mu_0 I}{2\pi d}
\]
Example: Infinite Wire

A cylindrical conductor has radius $R$ and a uniform current density with total current $I$.

Find $B$ everywhere.

Two regions:
(1) outside wire ($r \geq R$)
(2) inside wire ($r < R$)
Ampere’s Law Example: Infinite Wire

Amperian Loop: B is Constant & Parallel I Penetrates
Example: Wire of Radius $R$

Region 1: Outside wire ($r \geq R$)

Cylindrical symmetry $\rightarrow$

Amperian Circle

B-field counterclockwise

\[
\oint \vec{B} \cdot d\vec{s} = B \oint d\vec{s} = B \left( 2\pi r \right) = \mu_0 I_{enc} = \mu_0 I
\]

\[
\vec{B} = \frac{\mu_0 I}{2\pi r} \text{ counterclockwise}
\]
Example: Wire of Radius $R$

Region 2: Inside wire ($r < R$)

\[
\oint \mathbf{B} \cdot d\mathbf{s} = B \oint d\mathbf{s} = B \left( 2\pi r \right)
\]

\[
= \mu_0 I_{enc} = \mu_0 I \left( \frac{\pi r^2}{\pi R^2} \right)
\]

\[
\mathbf{B} = \frac{\mu_0 I r}{2\pi R^2} \text{ counterclockwise}
\]

Could also say: \( J = \frac{I}{A} = \frac{I}{\pi R^2} \); \( I_{enc} = J A_{enc} = \frac{I}{\pi R^2} \left( \pi r^2 \right) \)
OVERALL FIELD

\[ r \leq R \]

\[ 2\pi r B(r) = \mu_o I \left( \frac{r^2}{R^2} \right) \]

\[ B(r) = \left( \frac{\mu_o}{2\pi} \right) I \left( \frac{r}{R^2} \right) \]

\[ r \geq R \]

\[ 2\pi r B(r) = \mu_o I \]

\[ B(r) = \left( \frac{\mu_o}{2\pi} \right) \left( \frac{I}{r} \right) \]
Applying Ampere’s Law

In Choosing Amperian Loop:

• Study & Follow Symmetry
• Determine Field Directions First
• Think About Where Field is Zero
• Loop Must
  • Be Parallel to (Constant) Desired Field
  • Be Perpendicular to Unknown Fields
  • Or Be Located in Zero Field
What is a solenoid – A device that creates a uniform magnetic field inside and zero outside (in both cases almost uniform and almost zero)
Who needs it. Electronic devices, MRI machines, Fusion machines etc

The magnetic field is uniform inside this section of an ideal, infinitely long solenoid.
The magnetic field outside the solenoid is zero.
How to make a solenoid

(a) A single loop

(b) A stack of three loops

The fields of the three loops nearly cancel here.

The fields reinforce each other here.

loosely wound
tightly wound
Magnetic Field of Ideal Solenoid

Using Ampere’s law: Think!

\[ \oint \vec{B} \cdot d\vec{s} = \int B \cdot d\vec{s} + \int B \cdot d\vec{s} + \int B \cdot d\vec{s} + \int B \cdot d\vec{s} \]

\[ = Bl + 0 + 0 + 0 + 0 \]

\[ I_{enc} = nLI \quad n: \text{turn density} \]

\[ \oint \vec{B} \cdot d\vec{s} = Bl = \mu_0 nLI \]

\[ n = \frac{N}{L} : \# \text{turns/unit length} \]

\[ B = \frac{\mu_0 nLI}{l} = \mu_0 nI \]
B FIELD INSIDE A SOLENOID

The magnetic field is uniform inside this section of an ideal, infinitely long solenoid. The magnetic field outside the solenoid is zero.

\[ \oint \vec{B} \cdot d\vec{s} = Bl = \mu_0NI \]

uniform magnetic field inside a solenoid is

\[ B_{\text{solenoid}} = \frac{\mu_0NI}{l} = \mu_0nI \]

**FIGURE 33.30** A closed path inside and outside an ideal solenoid.

This is the integration path for Ampère’s law. There are \( N \) turns inside.

\( \vec{B} \) is tangent to the integration path along the bottom edge.

\( n \) is turns per unit length (e.g. per meter)