Tips for Ch. 33:

In this Chapter, we have shown that in the presence of both magnetic field $\vec{B}$ and the electric field $\vec{E}$, the total force acting on a moving particle with charge $q$ is $\vec{F} = \vec{F}_e + \vec{F}_b = q(\vec{E} + \vec{v} \times \vec{B})$, where $\vec{v}$ is the velocity of the particle. The direction of $\vec{F}_b$ involves the cross product of $\vec{v}$ and $\vec{B}$, based on the right-hand rule. In Cartesian coordinates, the unit vectors are $\hat{i}$, $\hat{j}$ and $\hat{k}$ which satisfy the following properties:

\[
\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}
\]

\[
\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}
\]

\[
\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0
\]

For $\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$, the cross product may be obtained as

\[
\vec{v} \times \vec{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
v_x & v_y & v_z \\
B_x & B_y & B_z
\end{vmatrix} = (v_yB_z - v_zB_y)\hat{i} + (v_zB_x - v_xB_z)\hat{j} + (v_xB_y - v_yB_x)\hat{k}
\]

If only the magnetic field is present, and $\vec{v}$ is perpendicular to $\vec{B}$, then the trajectory is a circle with a radius $r = mv/|q|B$, and an angular speed $\omega = |q|B/m$.

When dealing with a more complicated case, it is useful to work with individual force components. For example,

\[
F_x = ma_x = qE_x + q(v_yB_z - v_zB_y)
\]
Coordinate Systems and triple products of unit vectors:

\[ e_x \times e_y = e_z \]

\[ e_r \times e_\phi = e_z \]

\[ e_r \times e_\theta = e_\phi \]
8.9.1 Rolling Rod

A rod with a mass \( m \) and a radius \( R \) is mounted on two parallel rails of length \( a \) separated by a distance \( l \), as shown in the Figure 8.9.1. The rod carries a current \( I \) and rolls without slipping along the rails which are placed in a uniform magnetic field \( \vec{B} \) directed into the page. If the rod is initially at rest, what is its speed as it leaves the rails?

![Figure 8.9.1 Rolling rod in uniform magnetic field](image)

**Solution:**

Using the coordinate system shown on the right, the magnetic force acting on the rod is given by

\[
\vec{F}_m = I \vec{E} \times \vec{B} = I(\vec{E}) \times (-\vec{B}) = -I \vec{E} \hat{J}
\]  

(8.9.1)

The total work done by the magnetic force on the rod as it moves through the region is

\[
W = \int \vec{F}_m \cdot d\vec{s} = f_\text{p} a = (I \vec{E}) a
\]

(8.9.2)

By the work-energy theorem, \( W \) must be equal to the change in kinetic energy:

\[
\Delta E = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2
\]

(8.9.3)

where both translation and rolling are involved. Since the moment of inertia of the rod is given by \( I = \frac{mR^2}{2} \), and the condition of rolling without slipping implies \( \omega = \frac{v}{R} \), we have

\[
I \omega a = \frac{1}{2} mv^2 + \frac{1}{2} \left( \frac{mR^2}{2} \right) \left( \frac{v}{R} \right)^2 = \frac{1}{2} mv^2 + \frac{1}{4} mv^2 = \frac{3}{4} mv^2
\]

(8.9.4)

Thus, the speed of the rod as it leaves the rails is

\[
v = \sqrt{\frac{4 I \vec{E} a}{3m}}
\]

(8.9.5)
8.9.2 Suspended Conducting Rod

A conducting rod having a mass density $\lambda$ kg/m is suspended by two flexible wires in a uniform magnetic field $\mathbf{B}$ which points out of the page, as shown in Figure 8.9.2.

![Figure 8.9.2 Suspended conducting rod in uniform magnetic field](image)

If the tension on the wires is zero, what are the magnitude and the direction of the current in the rod?

**Solution:**

In order that the tension in the wires be zero, the magnetic force $\mathbf{F}_s = I \mathbf{\ell} \times \mathbf{B}$ acting on the conductor must exactly cancel the downward gravitational force $\mathbf{F}_g = -mg \mathbf{\hat{k}}$.

For $\mathbf{F}_s$ to point in the $+z$-direction, we must have $\mathbf{\ell} = -\mathbf{\hat{j}}$, i.e., the current flows to the left, so that

$$\mathbf{F}_s = I \mathbf{\ell} \times \mathbf{B} = I (\mathbf{\hat{z}} \times (B \mathbf{\hat{i}})) = -I \ell B \mathbf{\hat{j}} = +I \ell B \mathbf{\hat{k}}$$  \hspace{1cm} (8.9.6)

The magnitude of the current can be obtained from

$$I \ell B = mg$$  \hspace{1cm} (8.9.7)

or

$$I = \frac{mg}{\ell B} = \frac{\lambda g}{B}$$  \hspace{1cm} (8.9.8)
8.9.3 Charged Particles in Magnetic Field

Particle A with charge $q$ and mass $m_A$ and particle B with charge $2q$ and mass $m_B$, are accelerated from rest by a potential difference $\Delta V$, and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle $A$ and $B$ are $R$ and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particle. What is their mass ratio?

Solution:

The kinetic energy gained by the charges is equal to

$$\frac{1}{2}mv^2 = q\Delta V$$  \hspace{1cm} (8.9.9)

which yields

$$v = \sqrt{\frac{2q\Delta V}{m}}$$  \hspace{1cm} (8.9.10)

The charges move in semicircles, since the magnetic force points radially inward and provides the source of the centripetal force:

$$\frac{mv^2}{r} = qvB$$  \hspace{1cm} (8.9.11)

The radius of the circle can be readily obtained as:

$$r = \frac{mv}{qB} = \frac{m}{qB} \sqrt{\frac{2q\Delta V}{m}} = \frac{1}{B} \sqrt{\frac{2m\Delta V}{q}}$$  \hspace{1cm} (8.9.12)

which shows that $r$ is proportional to $(m/q)^{1/2}$. The mass ratio can then be obtained from

$$\frac{r_A}{r_B} = \left(\frac{m_A}{q_A}\right)^{1/2} \Rightarrow \frac{R}{2R} = \left(\frac{m_A}{2q}\right)^{1/2}$$

which gives

$$\frac{m_A}{m_B} = \frac{1}{8}$$  \hspace{1cm} (8.9.14)
8.9.4 Bar Magnet in Non-Uniform Magnetic Field

A bar magnet with its north pole up is placed along the symmetric axis below a horizontal conducting ring carrying current $I$, as shown in the Figure 8.9.3. At the location of the ring, the magnetic field makes an angle $\theta$ with the vertical. What is the force on the ring?

![Figure 8.9.3 A bar magnet approaching a conducting ring](image)

**Solution:**

The magnetic force acting on a small differential current-carrying element $ld\,\hat{s}$ on the ring is given by $d\vec{F} = I(d\hat{s}) \times \vec{B}$, where $\vec{B}$ is the magnetic field due to the bar magnet. Using cylindrical coordinates $(\hat{r}, \hat{\phi}, \hat{z})$ as shown in Figure 8.9.4, we have

$$d\vec{F} = I(-ds\,\hat{\phi}) \times (B\sin\theta\,\hat{r} + B\cos\theta\,\hat{z}) = (IBds)\sin\theta\,\hat{z} - (IBds)\cos\theta\,\hat{r} \quad (8.9.15)$$

Due to the axial symmetry, the radial component of the force will exactly cancel, and we are left with the $z$-component.

![Figure 8.9.4 Magnetic force acting on the conducting ring](image)

The total force acting on the ring then becomes

$$\vec{F}_z = (IB\sin\theta)\,\hat{z} \oint ds = (2\pi rIB\sin\theta)\,\hat{z} \quad (8.9.16)$$

The force points in the $+z$ direction and therefore is repulsive.
8.10 Conceptual Questions

1. Can a charged particle move through a uniform magnetic field without experiencing any force? Explain.

2. If no work can be done on a charged particle by the magnetic field, how can the motion of the particle be influenced by the presence of a field?

4. What type of magnetic field can exert a force on a magnetic dipole? Is the force repulsive or attractive?

5. If a compass needle is placed in a uniform magnetic field, is there a net magnetic force acting on the needle? Is there a net torque?

1. Yes if it moves parallel to B
2. Force perpendicular to v. Particle energy does not change only its direction. Similar to planetary motion around the Sun. Conservative system.
5. No force (see Fig.33,4)
9.10.1 Biot-Savart Law:

The law states that the magnetic field at a point $P$ due to a length element $d\mathbf{s}$ carrying a steady current $I$ located at $\mathbf{r}$ away is given by

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{s} \times \mathbf{r}}{r^3} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{s} \times \mathbf{\hat{r}}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

1. **Source point**: Choose an appropriate coordinate system and write down an expression for the differential current element $I \, d\mathbf{s}$, and the vector $\mathbf{\hat{r}}$ describing the position of $I \, d\mathbf{s}$. The magnitude $r' = |\mathbf{\hat{r}}|$ is the distance between $I \, d\mathbf{s}$ and the origin. Variables with a “prime” are used for the source point.

2. **Field point**: The field point $P$ is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\mathbf{r}_p$ for the field point $P$. The quantity $r_p = |\mathbf{r}_p|$ is the distance between the origin and $P$.

3. **Relative position vector**: The relative position between the source point and the field point is characterized by the relative position vector $\mathbf{\hat{r}} = \mathbf{\hat{r}}_p - \mathbf{\hat{r}}$. The corresponding unit vector is

$$\mathbf{\hat{r}} = \frac{\mathbf{\hat{r}}_p - \mathbf{\hat{r}}}{r} = \frac{|\mathbf{\hat{r}}_p - \mathbf{\hat{r}}|}{r}$$

where $r = |\mathbf{\hat{r}}| = |\mathbf{\hat{r}}_p - \mathbf{\hat{r}}|$ is the distance between the source and the field point $P$.

4. Calculate the cross product $d\mathbf{s} \times \mathbf{\hat{r}}$ or $d\mathbf{s} \times \mathbf{\hat{r}}$. The resultant vector gives the direction of the magnetic field $\mathbf{B}$, according to the Biot-Savart law.

5. Substitute the expressions obtained to $d\mathbf{B}$ and simplify as much as possible.

6. Complete the integration to obtain $\mathbf{B}$ if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.
9.10.2 Ampere's law:

Ampere's law states that the line integral of $\mathbf{B} \cdot d\mathbf{s}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{enc}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

1. Draw an Amperian loop using symmetry arguments.
2. Find the current enclosed by the Amperian loop.
3. Calculate the line integral $\oint \mathbf{B} \cdot d\mathbf{s}$ around the closed loop.
4. Equate $\oint \mathbf{B} \cdot d\mathbf{s}$ with $\mu_0 I_{enc}$ and solve for $\mathbf{B}$.
9.11.2 Current-Carrying Arc

Consider the current-carrying loop formed of radial lines and segments of circles whose centers are at point \( P \) as shown below. Find the magnetic field \( \vec{B} \) at \( P \).

![Figure 9.11.2 Current-carrying arc](image)

Solution:

According to the Biot-Savart law, the magnitude of the magnetic field due to a differential current-carrying element \( l \, d \vec{s} \) is given by

\[
dB = \frac{\mu_0 l}{4\pi} \frac{|d \vec{s} \times \vec{r}|}{r^2} = \frac{\mu_0 l}{4\pi} \frac{r \, d\theta'}{r^2} = \frac{\mu_0 l}{4\pi} \frac{d\theta'}{r}
\]  

\[(9.11.6)\]

For the outer arc, we have

\[
B_{outer} = \frac{\mu_0 l}{4\pi b} \int_0^\theta d\theta' = \frac{\mu_0 l \theta}{4\pi b}
\]  

\[(9.11.7)\]

The direction of \( \vec{B}_{outer} \) is determined by the cross product \( d \vec{s} \times \vec{r} \) which points out of the page. Similarly, for the inner arc, we have

\[
B_{inner} = \frac{\mu_0 l}{4\pi a} \int_0^\theta d\theta' = \frac{\mu_0 l \theta}{4\pi a}
\]  

\[(9.11.8)\]

For \( \vec{B}_{inner} \), \( d \vec{s} \times \vec{r} \) points into the page. Thus, the total magnitude of magnetic field is

\[
\vec{B} = \vec{B}_{inner} + \vec{B}_{outer} = \frac{\mu_0 l \theta}{4\pi} \left( \frac{1}{a} - \frac{1}{b} \right) \text{ (into page)}
\]  

\[(9.11.9)\]
9.11.4 Hairpin-Shaped Current-Carrying Wire

An infinitely long current-carrying wire is bent into a hairpin-like shape shown in Figure 9.11.4. Find the magnetic field at the point \( P \) which lies at the center of the half-circle.

![Hairpin-shaped current-carrying wire](image)

**Figure 9.11.4** Hairpin-shaped current-carrying wire

**Solution:**

Again we break the wire into three parts: two semi-infinite plus a semi-circular segments.

(i) Let \( P \) be located at the origin in the \( xy \) plane. The first semi-infinite segment then extends from \((x, y) = (-\infty, -r)\) to \((0, -r)\). The two angles which parameterize this segment are characterized by \( \cos \theta_1 = 1 (\theta_1 = 0) \) and \( \cos \theta_2 = 0 (\theta_2 = \pi / 2) \). Therefore, its contribution to the magnetic field at \( P \) is

\[
B_i = \frac{\mu_0 I}{4\pi r} (\cos \theta_1 + \cos \theta_2) = \frac{\mu_0 I}{4\pi r} (1 + 0) = \frac{\mu_0 I}{4\pi r}
\]

(9.11.16)

The direction of \( \mathbf{B}_i \) is out of page, or \( +\mathbf{k} \).
(ii) For the semi-circular arc of radius \( r \), we make use of the Biot-Savart law:

\[
\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{z} \times \hat{r}}{r^2}
\]

and obtain

\[
B_z = \frac{\mu_0 I}{4\pi} \int_0^\pi r d\theta \frac{\theta}{r^2} = \frac{\mu_0 I}{4r}
\]

(9.11.18)

The direction of \( \mathbf{B}_z \) is out of page, or \(+\hat{k}\).

(iii) The third segment of the wire runs from \((x, y) = (0, +r)\) to \((-\infty, +r)\). One may readily show that it gives the same contribution as the first one:

\[
B_3 = B_z = \frac{\mu_0 I}{4\pi r}
\]

(9.11.19)

The direction of \( \mathbf{B}_3 \) is again out of page, or \( +\hat{k} \).

The total magnitude of the magnetic field is

\[
\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 = 2\mathbf{B}_1 + \mathbf{B}_2 = \frac{\mu_0 I}{2\pi r} \hat{k} + \frac{\mu_0 I}{4r} \hat{k} = \frac{\mu_0 I}{4\pi r} (2 + \pi) \hat{k}
\]

(9.11.20)

Notice that the contribution from the two semi-infinite wires is equal to that due to an infinite wire:

\[
\mathbf{B}_1 + \mathbf{B}_3 = 2\mathbf{B}_1 = \frac{\mu_0 I}{2\pi r} \hat{k}
\]

(9.11.21)
46. a. Find an expression for the magnetic field at the center (point P) of the circular arc in FIGURE P33.46.
b. Does your result agree with the magnetic field of a current loop when $\theta = 2\pi$?

47. What are the strength and direction of the magnetic field at point P in FIGURE P33.47?
48. What is the magnetic field at the center of the loop in FIGURE P33.48?
**33.46. Model:** Use the Biot-Savart law for a current carrying segment.

**Visualize:** Please refer to Figure P33.46.

**Solve:** (a) The Biot-Savart law (Equation 33.6) for the magnetic field of a current segment $\Delta s$ is

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{I \Delta s \times \hat{r}}{r^3}$$

where the unit vector $\hat{r}$ points from current segment $\Delta s$ to the point, a distance $r$ away, at which we want to evaluate the field. For the two linear segments of the wire, $\Delta s$ is in the same direction as $\hat{r}$, so $\Delta s \times \hat{r} = \Delta s$. Thus

$$\vec{B} = \frac{\mu_0 I \Delta s}{4\pi r^3}$$

Now we are ready to sum the magnetic field of all the segments at point $P$. For all segments on the arc, the distance to point $P$ is $r = R$. The superposition of the fields is

$$\vec{B} = \frac{\mu_0 I}{4\pi R^3} \int ds \times \frac{\mu_0 L}{4\pi R^2} - \frac{\mu_0 I}{4\pi R}$$

where $L = R\theta$ is the length of the arc.

(b) Substituting $\theta = 2\pi$ in the above expression,

$$B_{\text{loop}} = \frac{\mu_0 I 2\pi}{4\pi R} = \frac{I}{2R}$$

This is Equation 33.7, which is the magnetic field at the center of a 1-turn coil.

**33.47. Model:** Use the Biot-Savart law for a current carrying segment.

**Visualize:** Please refer to Figure P33.47. The distance from $P$ to the inner arc is $r_1$ and the distance from $P$ to the outer arc is $r_2$.

**Solve:** As given in Equation 33.6, the Biot-Savart law for a current carrying small segment $\Delta s$ is

$$\vec{B} = \frac{\mu_0 I \Delta s \times \hat{r}}{4\pi r^3}$$

For the linear segments of the loop, $B_{21} = 0$ T because $\Delta s \times \hat{r} = 0$. Consider a segment $\Delta s$ on length on the inner arc. Because $\Delta s$ is perpendicular to the $\hat{r}$ vector, we have

$$\vec{B} = \frac{\mu_0 I \Delta s}{4\pi r_1^3} - \frac{\mu_0 I \Delta s}{4\pi r_2 r_1} \Rightarrow B_{\text{in}} = \int_{r_1}^{r_2} \frac{\mu_0 I \Delta s}{4\pi r} = \frac{\mu_0 I}{4\pi r_1} - \frac{\mu_0 I}{4\pi r_2}$$

A similar expression applies for $B_{\text{out}}$. The right-hand rule indicates an out-of-page direction for $B_{\text{in}}$, and an in-page direction for $B_{\text{out}}$. Thus,

$$\vec{B} = \left( \frac{\mu_0 I}{4r_1}, \text{out of page} \right) - \left( \frac{\mu_0 I}{4r_2}, \text{in to page} \right) = \left[ \frac{\mu_0 I}{4}, \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \right], \text{into page}$$

The field strength is

$$B = \left(4\pi \times 10^{-7} \text{ T m/A} \right) \left( \frac{1}{0.010 \text{ m}} - \frac{1}{0.020 \text{ m}} \right) = 7.9 \times 10^{-7} \text{ T}$$

Thus $\vec{B} = (7.9 \times 10^{-7} \text{ T}, \text{ into page})$. 

Solutions 46,47,48
33.48. Model: Assume that the wire is infinitely long.

Visualize: Please refer to Figure P33.48. The wire, looped as it is, consists of a circular part and a linear part.

Solve: Using Equation 33.7 and Example 33.3, the magnetic field at P is

\[ B_P = B_{\text{loop center}} + B_{\text{wire}} = \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{2\pi R} \]

\[ = \frac{4\pi \left( 10^{-7} \text{ T m/A} \right) \left( 5.0 \text{ A} \right)}{2 \left( 0.010 \text{ m} \right)} + \frac{4\pi \left( 10^{-7} \text{ T m/A} \right) \left( 5.0 \text{ A} \right)}{2\pi \left( 0.010 \text{ m} \right)} = 4.1 \times 10^{-4} \text{ T} \]