PHYS 270
Spring 2011

Solved Problems

Chapters 33, 34. 35

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Tips for Ch. 33:

In this Chapter, we have shown that in the presence of both magnetic field $\vec{\mathbf{B}}$ and the electric field $\vec{\mathbf{E}}$, the total force acting on a moving particle with charge q is $\vec{\mathbf{F}} = \vec{\mathbf{F}}_e + \vec{\mathbf{F}}_B = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$, where $\vec{\mathbf{v}}$ is the velocity of the particle. The direction of $\vec{\mathbf{F}}_B$ involves the cross product of $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, based on the right-hand rule. In Cartesian coordinates, the unit vectors are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ which satisfy the following properties:

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$

For $\vec{\mathbf{v}} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$, the cross product may be obtained as

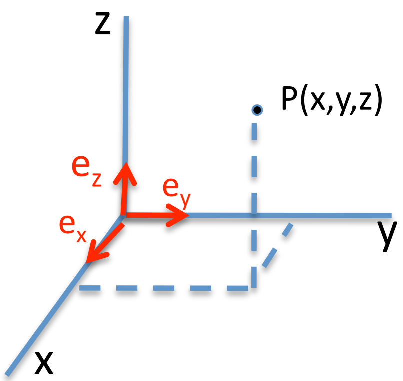
$$\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = (v_y B_z - v_z B_y)\hat{\mathbf{i}} + (v_z B_x - v_x B_z)\hat{\mathbf{j}} + (v_x B_y - v_y B_x)\hat{\mathbf{k}}$$

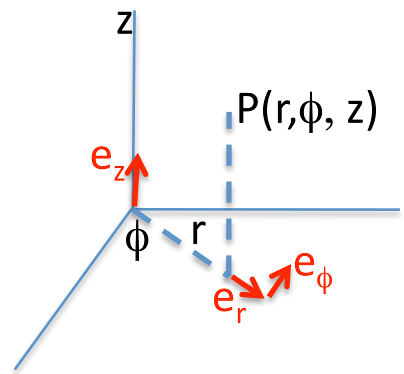
If only the magnetic field is present, and $\vec{\mathbf{v}}$ is perpendicular to $\vec{\mathbf{B}}$, then the trajectory is a circle with a radius $r = mv / |q| B$, and an angular speed $\omega = |q| B / m$.

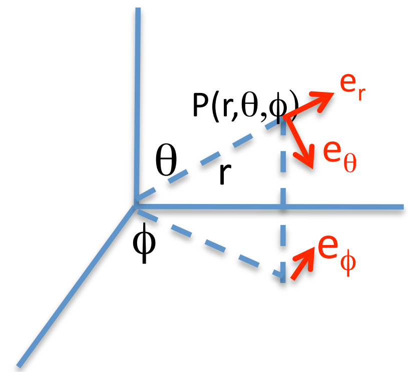
When dealing with a more complicated case, it is useful to work with individual force components. For example,

$$F_x = ma_x = qE_x + q(v_y B_z - v_z B_y)$$

Coordinate Systems and triple products of unit vectors:







$e_x \times e_y = e_z$

$e_r \times e_\phi = e_z$

$e_r \times e_\theta = e_\phi$

8.9.1 Rolling Rod

A rod with a mass m and a radius R is mounted on two parallel rails of length a separated by a distance ℓ, as shown in the Figure 8.9.1. The rod carries a current I and rolls without slipping along the rails which are placed in a uniform magnetic field $\vec{\mathbf{B}}$ directed into the page. If the rod is initially at rest, what is its speed as it leaves the rails?

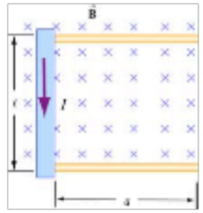
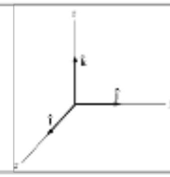


Figure 8.9.1 Rolling rod in uniform magnetic field

Solution:

Using the coordinate system shown on the right, the magnetic force acting on the rod is given by

$$\vec{\mathbf{F}}_B = I\vec{\ell}\times\vec{\mathbf{B}} = I(\ell\,\hat{\mathbf{i}})\times(-B\,\hat{\mathbf{k}}) = I\ell B\,\hat{\mathbf{j}} \qquad (8.9.1)$$



The total work done by the magnetic force on the rod as it moves through the region is

$$W = \int \vec{\mathbf{F}}_B \cdot d\,\vec{\mathbf{s}} = F_B a = (I\ell B)a \qquad (8.9.2)$$

By the work-energy theorem, W must be equal to the change in kinetic energy:

$$\Delta K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \qquad (8.9.3)$$

where both translation and rolling are involved. Since the moment of inertia of the rod is given by $I = mR^2/2$, and the condition of rolling with slipping implies $\omega = v/R$, we have

$$I\ell Ba = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2 \qquad (8.9.4)$$

Thus, the speed of the rod as it leaves the rails is

$$v = \sqrt{\frac{4I\ell Ba}{3m}} \qquad (8.9.5)$$

8.9.2 Suspended Conducting Rod

A conducting rod having a mass density λ kg/m is suspended by two flexible wires in a uniform magnetic field $\vec{\mathbf{B}}$ which points out of the page, as shown in Figure 8.9.2.

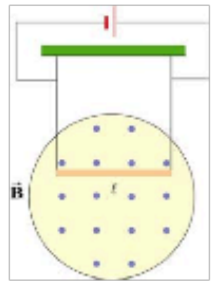
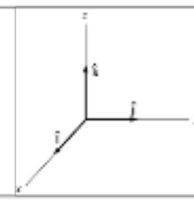


Figure 8.9.2 Suspended conducting rod in uniform magnetic field

If the tension on the wires is zero, what are the magnitude and the direction of the current in the rod?

Solution:

In order that the tension in the wires be zero, the magnetic force $\vec{\mathbf{F}}_B = I\vec{\ell}\times\vec{\mathbf{B}}$ acting on the conductor must exactly cancel the downward gravitational force $\vec{\mathbf{F}}_g = -mg\hat{\mathbf{k}}$.



For $\vec{\mathbf{F}}_B$ to point in the $+z$-direction, we must have $\vec{\ell} = -\ell\,\hat{\mathbf{j}}$, i.e., the current flows to the left, so that

$$\vec{\mathbf{F}}_B = I\vec{\ell}\times\vec{\mathbf{B}} = I(-\ell\,\hat{\mathbf{j}})\times(B\,\hat{\mathbf{i}}) = -I\ell B(\hat{\mathbf{j}}\times\hat{\mathbf{i}}) = +I\ell B\,\hat{\mathbf{k}} \tag{8.9.6}$$

The magnitude of the current can be obtain from

$$I\ell B = mg \tag{8.9.7}$$

or

$$I = \frac{mg}{B\ell} = \frac{\lambda g}{B} \tag{8.9.8}$$

8.9.3 Charged Particles in Magnetic Field

Particle A with charge q and mass m_A and particle B with charge $2q$ and mass m_B, are accelerated from rest by a potential difference ΔV, and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle A and B are R and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particle. What is their mass ratio?

Solution:

The kinetic energy gained by the charges is equal to

$$\frac{1}{2}mv^2 = q\Delta V \tag{8.9.9}$$

which yields

$$v = \sqrt{\frac{2q\Delta V}{m}} \tag{8.9.10}$$

The charges move in semicircles, since the magnetic force points radially inward and provides the source of the centripetal force:

$$\frac{mv^2}{r} = qvB \tag{8.9.11}$$

The radius of the circle can be readily obtained as:

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2q\Delta V}{m}} = \frac{1}{B}\sqrt{\frac{2m\Delta V}{q}} \tag{8.9.12}$$

which shows that r is proportional to $(m/q)^{1/2}$. The mass ratio can then be obtained from

$$\frac{r_A}{r_B} = \frac{(m_A/q_A)^{1/2}}{(m_B/q_B)^{1/2}} \quad \Rightarrow \quad \frac{R}{2R} = \frac{(m_A/q)^{1/2}}{(m_B/2q)^{1/2}} \tag{8.9.13}$$

which gives

$$\frac{m_A}{m_B} = \frac{1}{8} \tag{8.9.14}$$

8.9.4 Bar Magnet in Non-Uniform Magnetic Field

A bar magnet with its north pole up is placed along the symmetric axis below a horizontal conducting ring carrying current I, as shown in the Figure 8.9.3. At the location of the ring, the magnetic field makes an angle θ with the vertical. What is the force on the ring?

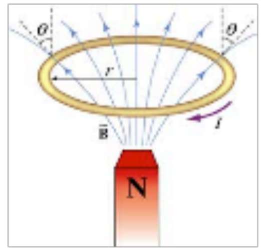


Figure 8.9.3 A bar magnet approaching a conducting ring

Solution:

The magnetic force acting on a small differential current-carrying element $Id\vec{\mathbf{s}}$ on the ring is given by $d\vec{\mathbf{F}}_B = Id\vec{\mathbf{s}}\times\vec{\mathbf{B}}$, where $\vec{\mathbf{B}}$ is the magnetic field due to the bar magnet. Using cylindrical coordinates $(\hat{\mathbf{r}},\hat{\boldsymbol{\varphi}},\hat{\mathbf{z}})$ as shown in Figure 8.9.4, we have

$$d\vec{\mathbf{F}}_B = I(-ds\,\hat{\boldsymbol{\varphi}})\times(B\sin\theta\,\hat{\mathbf{r}} + B\cos\theta\,\hat{\mathbf{z}}) = (IBds)\sin\theta\,\hat{\mathbf{z}} - (IBds)\cos\theta\,\hat{\mathbf{r}} \quad (8.9.15)$$

Due to the axial symmetry, the radial component of the force will exactly cancel, and we are left with the z-component.

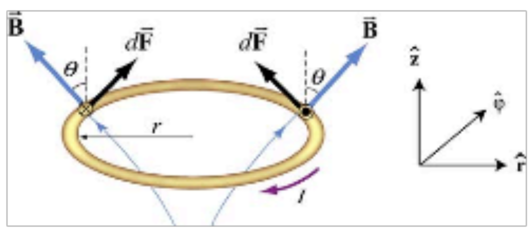


Figure 8.9.4 Magnetic force acting on the conducting ring

The total force acting on the ring then becomes

$$\vec{\mathbf{F}}_B = (IB\sin\theta)\hat{\mathbf{z}}\oint ds = (2\pi rIB\sin\theta)\hat{\mathbf{z}} \quad (8.9.16)$$

The force points in the $+z$ direction and therefore is repulsive.

8.10 Conceptual Questions

1. Can a charged particle move through a uniform magnetic field without experiencing any force? Explain.

2. If no work can be done on a charged particle by the magnetic field, how can the motion of the particle be influenced by the presence of a field?

4. What type of magnetic field can exert a force on a magnetic dipole? Is the force repulsive or attractive?

5. If a compass needle is placed in a uniform magnetic field, is there a net magnetic force acting on the needle? Is there a net torque?

1. Yes if it moves parallel to B
2. Force perpendicular to v. Particle energy does not change only its direction. Similar to planetary motion around the Sun. Conservative system.

4. Non-uniform. Can be either. See previous problem.

5. No force (see Fig.33,4)

Biot-Savart and Ampere Problems - Strategy

9.10.1 Biot-Savart Law:

The law states that the magnetic field at a point P due to a length element $d\vec{\mathbf{s}}$ carrying a steady current I located at $\vec{\mathbf{r}}$ away is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $I\,d\vec{\mathbf{s}}$, and the vector $\vec{\mathbf{r}}\,'$ describing the position of $I\,d\vec{\mathbf{s}}$. The magnitude $r' = |\vec{\mathbf{r}}\,'|$ is the distance between $I\,d\vec{\mathbf{s}}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point P is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\vec{\mathbf{r}}_P$ for the field point P. The quantity $r_P = |\vec{\mathbf{r}}_P|$ is the distance between the origin and P.

(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|}$$

where $r = |\vec{\mathbf{r}}| = |\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|$ is the distance between the source and the field point P.

(4) Calculate the cross product $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$ or $d\vec{\mathbf{s}}\times\vec{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field $\vec{\mathbf{B}}$, according to the Biot-Savart law.

(5) Substitute the expressions obtained to $d\vec{\mathbf{B}}$ and simplify as much as possible.
(6) Complete the integration to obtain Bif possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

9.10.2 Ampere's law:

Ampere's law states that the line integral of $\vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \mu_0 I_{enc}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

(1) Draw an Amperian loop using symmetry arguments.

(2) Find the current enclosed by the Amperian loop.

(3) Calculate the line integral $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around the closed loop.

(4) Equate $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ with $\mu_0 I_{enc}$ and solve for $\vec{\mathbf{B}}$.

9.11.2 Current-Carrying Arc

Consider the current-carrying loop formed of radial lines and segments of circles whose centers are at point P as shown below. Find the magnetic field $\vec{\mathbf{B}}$ at P.

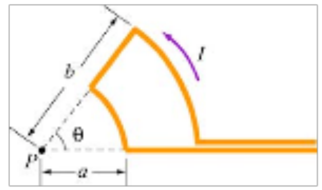


Figure 9.11.2 Current-carrying arc

Solution:

According to the Biot-Savart law, the magnitude of the magnetic field due to a differential current-carrying element $I\,d\vec{\mathbf{s}}$ is given by

$$dB = \frac{\mu_0 I}{4\pi}\frac{|d\vec{\mathbf{s}}\times\hat{\mathbf{r}}|}{r^2} = \frac{\mu_0 I}{4\pi}\frac{r\,d\theta'}{r^2} = \frac{\mu_0 I}{4\pi r}d\theta' \tag{9.11.6}$$

For the outer arc, we have

$$B_{\text{outer}} = \frac{\mu_0 I}{4\pi b}\int_0^\theta d\theta' = \frac{\mu_0 I\theta}{4\pi b} \tag{9.11.7}$$

The direction of $\vec{\mathbf{B}}_{\text{outer}}$ is determined by the cross product $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$ which points out of the page. Similarly, for the inner arc, we have

$$B_{\text{inner}} = \frac{\mu_0 I}{4\pi a}\int_0^\theta d\theta' = \frac{\mu_0 I\theta}{4\pi a} \tag{9.11.8}$$

For $\vec{\mathbf{B}}_{\text{inner}}$, $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$ points into the page. Thus, the total magnitude of magnetic field is

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_{\text{inner}} + \vec{\mathbf{B}}_{\text{outer}} = \frac{\mu_0 I\theta}{4\pi}\left(\frac{1}{a}-\frac{1}{b}\right) \text{ (into page)} \tag{9.11.9}$$

9.11.4 Hairpin-Shaped Current-Carrying Wire

An infinitely long current-carrying wire is bent into a hairpin-like shape shown in Figure 9.11.4. Find the magnetic field at the point P which lies at the center of the half-circle.

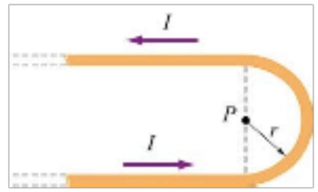


Figure 9.11.4 Hairpin-shaped current-carrying wire

Solution:

Again we break the wire into three parts: two semi-infinite plus a semi-circular segments.

(i) Let P be located at the origin in the xy plane. The first semi-infinite segment then extends from $(x, y) = (-\infty, -r)$ to $(0, -r)$. The two angles which parameterize this segment are characterized by $\cos\theta_1 = 1\,(\theta_1 = 0)$ and $\cos\theta_2 = 0\ (\theta_2 = \pi/2)$. Therefore, its contribution to the magnetic field at P is

$$B_1 = \frac{\mu_0 I}{4\pi r}\left(\cos\theta_1 + \cos\theta_2\right) = \frac{\mu_0 I}{4\pi r}(1+0) = \frac{\mu_0 I}{4\pi r} \tag{9.11.16}$$

The direction of $\vec{\mathbf{B}}_1$ is out of page, or $+\hat{\mathbf{k}}$.

(ii) For the semi-circular arc of radius r, we make use of the Biot-Savart law:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{9.11.17}$$

and obtain

$$B_2 = \frac{\mu_0 I}{4\pi} \int_0^{\pi} \frac{r d\theta}{r^2} = \frac{\mu_0 I}{4r} \tag{9.11.18}$$

The direction of $\vec{\mathbf{B}}_2$ is out of page, or $+\hat{\mathbf{k}}$.

(iii) The third segment of the wire runs from $(x, y) = (0, +r)$ to $(-\infty, +r)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 = \frac{\mu_0 I}{4\pi r} \tag{9.11.19}$$

The direction of $\vec{\mathbf{B}}_3$ is again out of page, or $+\hat{\mathbf{k}}$.

The total magnitude of the magnetic field is

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{k}} + \frac{\mu_0 I}{4r}\hat{\mathbf{k}} = \frac{\mu_0 I}{4\pi r}(2+\pi)\hat{\mathbf{k}} \tag{9.11.20}$$

Notice that the contribution from the two semi-infinite wires is equal to that due to an infinite wire:

$$\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{k}} \tag{9.11.21}$$

46. ‖ a. Find an expression for the magnetic field at the center (point P) of the circular arc in FIGURE P33.46.
b. Does your result agree with the magnetic field of a current loop when $\theta = 2\pi$?

FIGURE P33.46

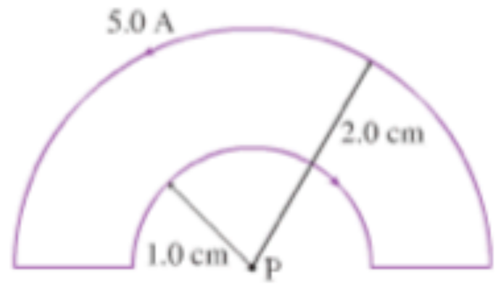


FIGURE P33.47

47. ‖ What are the strength and direction of the magnetic field at point P in FIGURE P33.47?

48. ‖ What is the magnetic field at the center of the loop in FIGURE P33.48?

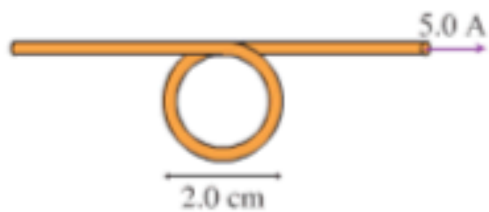


FIGURE P33.48

Solutions 46,47,48

33.46. Model: Use the Biot-Savart law for a current carrying segment.
Visualize: Please refer to Figure P33.46.
Solve: **(a)** The Biot-Savart law (Equation 33.6) for the magnetic field of a current segment $\Delta\vec{s}$ is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\vec{s}\times\hat{r}}{r^2}$$

where the unit vector $\hat{r}$ points from current segment Δs to the point, a distance r away, at which we want to evaluate the field. For the two linear segments of the wire, $\Delta\vec{s}$ is in the same direction as $\hat{r}$, so $\Delta\vec{s}\times\hat{r} = 0$. For the curved segment, $\Delta\vec{s}$ and $\hat{r}$ are always perpendicular, so $\Delta\vec{s}\times\hat{r} = \Delta s$. Thus

$$B = \frac{\mu_0}{4\pi}\frac{I\Delta s}{r^2}$$

Now we are ready to sum the magnetic field of all the segments at point P. For all segments on the arc, the distance to point P is $r = R$. The superposition of the fields is

$$B = \frac{\mu_0}{4\pi}\frac{I}{R^2}\int_{\text{arc}} ds = \frac{\mu_0}{4\pi}\frac{IL}{R^2} = \frac{\mu_0 I\theta}{4\pi R}$$

where $L = R\theta$ is the length of the arc.
(b) Substituting $\theta = 2\pi$ in the above expression,

$$B_{\text{loop center}} = \frac{\mu_0 I 2\pi}{4\pi R} = \frac{\mu_0}{2}\frac{I}{R}$$

This is Equation 33.7, which is the magnetic field at the center of a 1-turn coil.

33.47. Model: Use the Biot-Savart law for a current carrying segment.
Visualize: Please refer to Figure P33.47. The distance from P to the inner arc is r_1 and the distance from P to the outer arc is r_2.
Solve: As given in Equation 33.6, the Biot-Savart law for a current carrying small segment $\Delta\vec{s}$ is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\vec{s}\times\hat{r}}{r^2}$$

For the linear segments of the loop, $B_{\Delta s} = 0$ T because $\Delta\vec{s}\times\hat{r} = 0$. Consider a segment $\Delta\vec{s}$ on length on the inner arc. Because $\Delta\vec{s}$ is perpendicular to the $\hat{r}$ vector, we have

$$B = \frac{\mu_0}{4\pi}\frac{I\Delta s}{r_1^2} = \frac{\mu_0}{4\pi}\frac{Ir_1\Delta\theta}{r_1^2} = \frac{\mu_0}{4\pi}\frac{I\Delta\theta}{r_1} \Rightarrow B_{\text{arc 1}} = \int_{-\pi/2}^{\pi/2}\frac{\mu_0 Id\theta}{4\pi r_1} = \frac{\mu_0 I}{4\pi r_1}\pi = \frac{\mu_0 I}{4r_1}$$

A similar expression applies for $B_{\text{arc 2}}$. The right-hand rule indicates an out-of-page direction for $B_{\text{arc 2}}$ and an into-page direction for $B_{\text{arc 1}}$. Thus,

$$\vec{B} = \left(\frac{\mu_0 I}{4r_1}, \text{ into page}\right) + \left(\frac{\mu_0 I}{4r_2}, \text{ out of page}\right) = \left[\frac{\mu_0 I}{4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \text{ into page}\right]$$

The field strength is

$$B = \frac{(4\pi\times10^{-7}\text{ T m/A})(5.0\text{ A})}{4}\left(\frac{1}{0.010\text{ m}} - \frac{1}{0.020\text{ m}}\right) = 7.9\times10^{-5}\text{ T}$$

Thus $\vec{B} = (7.9\times10^{-5}\text{ T, into page})$.

33.48. Model: Assume that the wire is infinitely long.

Visualize: Please refer to Figure P33.48. The wire, looped as it is, consists of a circular part and a linear part.

Solve: Using Equation 33.7 and Example 33.3, the magnetic field at P is

$$B_P = B_{\text{loop center}} + B_{\text{wire}} = \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{2\pi R}$$

$$= \frac{4\pi\left(10^{-7}\text{ T m/A}\right)\left(5.0\text{ A}\right)}{2\left(0.010\text{ m}\right)} + \frac{4\pi\left(10^{-7}\text{ T m/A}\right)\left(5.0\text{ A}\right)}{2\pi\left(0.010\text{ m}\right)} = 4.1\times10^{-4}\text{ T}$$